MODELLING THE DEPLETION OF RESOURCES AND THEIR CONSERVATION: EFFECTS OF CHANGES IN ENVIRONMENT AND HABITATS

by

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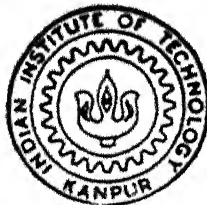**DEPARTMENT OF MATHEMATICS**

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MODELLING THE DEPLETION OF RESOURCES AND THEIR CONSERVATION. EFFECTS OF CHANGES IN ENVIRONMENT AND HABITATS

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BALRAM DUBEY

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Dedicated to

My

Mother Smt. Raj Kumari Devi

and

Father Shri Rama Shankar Dubey

CERTIFICATE

This is to certify that the matter embodied in the the entitled "MODELLING THE DEPLETION OF RESOURCES AND TH CONSERVATION : EFFECTS OF CHANGES. IN ENVIRONMENT AND HABITATS" Mr. Balram Dubey for the award of Degree of Doctor of Philoso of the Indian Institute of Technology Kanpur is a record bonafide research work carried out by him under my supervision guidance. The results embodied in this thesis have not b submitted to any other University or Institute for the award any degree or diploma.

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CHAPTER I

GENERAL INTRODUCTION

1.0   INTRODUCTION

It is well known that the resource carrying capacity of our planet is limited. Therefore the growth and development in various sectors of economy can not go on and on without depletion of resources and deterioration of the environment. The rapid industrialization, rising population and increasing energy requirements have stressed our environment and ecosystem considerably leading to various kinds of undesirable consequences causing great concern to mankind. The World Commission on Environment and Development (popularly known as Brundtland Commission) in its report on "Our Common Future" emphasized the need for sustainable development which is defined as the development that meets the needs of present generation without compromising the ability of the future generations to meet their own needs. From the discussions at the Earth Summit meet at RIO in June 1992 and agenda 21, it has become clear that the present generation must follow the path of sustainable development with appropriate use of resources causing little damage to environment and ecology. This can be achieved only by developing environment friendly technology for industrialization, methods for control of pollution, conservation of both renewable and non-renewable resources, improvement in consumer behavior and control of population particularly in the third world countries.

One of the important problems that the modern society faces today is the pollution of our environment affecting the quality of

life in the form of diseases, epidemics, etc. The abnormal level of green house gases in the atmosphere is affecting the climate, which has already changed to a considerable extent due to deforestation and man made projects, bringing prolonged drought, abnormal temperature in one region and occurrence of floods in the other, Treshow(1968), Woodwell(1970), Davis(1972), Maugh(1979), Smith(1981), Reish et al.(1982), Reish et. al. (1983), Parry and Carter(1988), Veeman(1988), Woodman and Cowling (1987).

Another menace to the society is the depletion of resources such as forestry, fisheries, fertile topsoil, crude oil, minerals, etc. These resources are being depleted due to rapid industrialization, fast urbanization and rising population damaging our ecology and environment to such an extent that if concrete steps are not taken soon to conserve these resources, many undesirable effects would occur leading to disastrous consequences for the mankind, Gadgil(1922), Stebbing(1922-27), Frevert et al.(1962), Detwyler(1971), Smith(1972), Pimental et al.(1976), Anon. (1977a,b), Das(1977), Gadgil and Prasad(1978), Karamchandani (1980), Brown(1981), Gadgil et al.(1983), Larson et al.(1983), Repetto and Holmes(1983), Brown and Wolf(1984), Haigh(1984), Gadgil(1985, 1987), Waring(1985), Biswas and Biswas(1986), Khoshoo(1986), Munn and Fedorov(1986), Shukla et al.(1987), Shukla et al.(1988), Gadgil and Chandran(1989), Shukla et al.(1989).

It is, therefore, absolutely essential to study the effects of various factors such as industrialization, pollution and population responsible for the depletion of resources so that appropriate measures for conservation are taken and the desired level of the resource biomass can be maintained without harming

our ecology and environment as well as the interests of future generations, Ghosh and Lohani(1972), Pathak(1974), Das(1977), Karamchandani(1980), Martino(1983), Khoshoo(1986), Lamberson (1986), Munn and Fedorov (1986), Shukla et al.(1987), Shukla et al.(1988), Shukla et al. (1989).

The objective of this thesis mainly is to study the problem of resource depletion caused by changes in environment and habitats and survival of species under such conditions. Specifically the effects of following factors on resource depletion have been studied by using appropriate mathematical models.

(i) industrialization, (ii) toxicant(pollutant) and (iii) population

Also in each case, model for the resource conservation by controlling industrialization, pollution or population has also been proposed and analysed.

Specifically the following two types of problems have been proposed and analysed in this thesis using mathematical models.

(i) Depletion and conservation of forestry resources : Effects of industrialization, pollution or Population

(ii) Effect of changing habitat on the survival of wildlife species

These models have been proposed by keeping in view the depletion of forestry resources caused by increasing industrialization, pollution or population in Doon Valley in Uttar Pradesh and Keoladeo National Wildlife Park in Bharatpur in Rajasthan in India and are applicable to situations around the world having similar ecology, Munn and Fedorov(1986), Ali and Vijayan(1986).

In the following we outline the relevant literature so that the research work carried out in the thesis related to above mentioned problems can be seen in its proper perspective.

1.1 DEPLETION AND CONSERVATION OF FORESTRY RESOURCES: EFFECTS OF INDUSTRIALIZATION, POLLUTION OR POPULATION

Since forests constitute an integral part of the biosphere, its depletion at an alarming rate due to rapid industrialization, over population, expansion of agricultural land, cutting of trees for fuel, fodder and paper based industries, etc. has caused many undesirable consequences. In particular, the depletion of forest has caused shortage of rainfall, lowering of soft water table, frequent drought, etc. leading to soil erosion, topsoil degradation, formation of wasteland and desertification. There are many ecologically unstable region around the world such as uplands of Western Amazonia, the Atlantic Coast of Brazil, the Madagascar Islands, the Himalayan foothills, the Malaysian rainforest zones, etc., Wilson(1989). In particular, the Doon Valley, in the northern part of Uttar Pradesh in India, is also an example where the main reasons for the depletion of forestry resources and threat to ecological stability are the following factors, Munn and Fedorov(1986).

1.1.1 LIMESTONE QUARRYING, INDUSTRIALIZATION AND ASSOCIATED POLLUTION

A major cause for the deterioration of environmental quality in the Doon Valley is rapid increase in limestone quarrying, industrialization and associated pollution. For example, in 1911 there were only four limestone quarriers in the Doon Valley and

during 1980 there were about 100 quarry leaseholderes in the Valley extracting 513 x 10^3 and 136 x 10^3 metric tones of lime stones and marble respectively. Limestone mining, cement plants, chemical plants, several small chimneys in sugar processing plants emitting black smoke are causing air pollutions. There are many non industrial sources such as vikram (a locally made type of taxi), domestic cooking stoves, outdoor fires, etc. causing air pollution. Some of the pollutants in the form of particular matter cover the leaves of trees and plants and thus inhibiting the photosynthesis process causing decrease in the growth rate of plants and trees.

1.1.2 GROWTH OF HUMAN POPULATIONS

The increase in human population is another cause for ecological degradation in the Valley. For instance there were 144000 human populations in 1880 in the Doon district and in 1980 there were 758000 human populations. The increase in the populations demands more land for agriculture and colonization, fuel wood, construction wood, limebricks, milk, meat and other commodities. The needs of urban market encouraged expansion of arable land, clearing of forests and other vegetation in the district. In fact, between 1880 and 1980 the area of arable land has expanded more than 41000 ha to 64000 ha. The demand for these resources has depleted the forestry resource to a very undesired level.

1.1.3 GROWTH OF LIVESTOCK POPULATIONS

The increase of livestock populations from 168000 to 315000 between 1890 to 1966 added additional pressure on forests and grasslands in the Valley.

For this Valley, an environmental programme under the auspices of International Institute for Applied Systems Analysis, Laxenburg(Austria) was started and a report was prepared in 1986 in which the above causes were mentioned for environmental and ecological degradation. A great emphasis was laid to study the problem of resource depletion using mathematical models, Munn and Fedorov(1986). In addition to the Doon Valley, several other zones also exists in India where the depletion of forestry resource has reached such a critical stage, Gadgil(1922), Anon.(1977a,b), Government of India(1976), Singh and Saxena(1980), Gadgil(1987) and mathematical modelling studies are needed.

1.1.4 MODELLING THE EFFECTS OF INDUSTRIALIZATION, POLLUTION OR POPULATION ON RESOURCE

Several investigations hevr been conducted to study the effects of industrialization,pollution or population on resource using mathematical models. Shukla et al.(1989) proposed a model to study the cumulative effect of industrialization and population on resource depletion and have shown that if the pressures of industrialization and population increase without control, the resource may not last long. However, if appropriate measures for conservation are taken, the resource can be maintained at a desired level even under the sustained pressure of industrialization and population.

Some investigations have also been made to study the effect of pollutants on biological species (resource) using mathematical models, Hallam and Clark(1982), Hallam et al. (1983a,b), Hallam and De Luna(1984), De Luna and Hallam(1987), Freedman and Shukla(1991), Huaping and Ma(1991). In particular, Hallam et al.(1983b) studied the effects of a toxic pollutant on a directly

exposed exposed population using a mathematical model of toxicant-population interaction. Hallam and de Luna(1984) further proposed a model and discussed the effects of a toxicant on a population when exposure is via environmental and food chain pathways. Their main mathematical results focused on effects of the toxicant on a population by indicating persistence and extinction criteria. De Luna and Hallam(1987) also proposed a mathematical model to study the effect of a toxicant on population and they showed that if the population exhibits a potential for growth and if there is a input of resource, then the population will persist. Huaping and Ma (1991) proposed a mathematical model to study the effects of toxicants on naturally stable two species communities. They studied the persistence-extinction thresholds for populations in toxicant stressed Lotka-Voleterra model of two interacting species. But in their model, the effect of toxicant on the carrying capacity of population has not been considered. In the work of Hallam et al.(1983), Hallam and De Luna(1984), De Luna and Hallam(1987), the growth rate of population density depends linearly upon the concentration of toxicant in the population and they also have not considered the effect of environmental concentration of toxicant on the carrying capacity of the population.

It may be pointed out here that in the above the concentration of toxicant was defined with respect to the biomass of the total population. Freedman and Shukla (1991), however, felt that if the biomass of the population, toxicant in the population and toxicant in the environment are defined with respect to mass or volume of the total environment in which the population lives, the model of ecotoxicological problems becomes more visible.

Keeping this in view, Freedman and Shukla (1991) proposed models to study the effect of a single toxicant on single species and predator prey systems. In case of single species growth they found some local and global dynamics and in case of predator-prey systems, they determined the existence of steady states for a small constant influx of toxicant.

In above investigations the depletion of biomass due to combined effect of industrialization, population or pollution have not been studied.

Keeping in view of the above literature survey, the following types of problems in the forthcoming chapters have been discussed (see summary of the thesis and corresponding chapters for details).

The chapter II of this thesis is focused to study the depletion of forestry resource due to combined effect of industrialization and pollution. In this chapter a conservation model is also proposed and analysed.

The chapter III of this thesis is devoted to study the depletion of forestry resource due to combined effect of industrialization and population. In this chapter a conservation model is also developed to conserve the resource biomass and to control the undesired level of industrialization and population.

The chapter IV of this thesis deals with the depletion of forestry resource due to combined effect of population and pollution. In this chapter a conservation model is also incorporated.

In chapter V of this thesis, a mathematical model is proposed and analysed to study the simultaneous effects of two toxicants on resource. A model to conserve the resource and to control the

toxicants is also proposed and analysed.

In chapter VI of this thesis, a mathematical model for soil depletion and its subsequent conservation is presented.

In chapter VII of this thesis, a mathematical model to study the effects of two interacting populations on the depletion of resource is proposed and analysed. A conservation model is also proposed.

1.2   EFFECT OF CHANGING HABITAT ON SURVIVAL OF SPECIES

It is well known that the ecosystems undergo degradation due to internal and external environmental influences causing significant changes in its structure and function, both abiotic and biotic(air, water, light, heat, plants, micro-organisms , animals, etc.). These changes may affect the carrying capacity of the habitat and consequently the biodiversity of the species. The habitat changes are caused by many factors but we have studied only the following effects on species.

1.2.1   DEPLETION OF RESOURCES IN A FORESTED HABITAT

As pointed earlier, forests play a pivotal role in maintaining the environment including atmospheric stability and in providing essential requirements to wildlife species living in this habitat. The depletion of forest biomass in the Doon Valley due to increase in human and cattle population and industrial development has not only caused damage to over all ecological structure of the Valley but also decreased the growth rate of resource dependent wildlife species in some cases and their extinction in several others, Munn and Fedorov(1986), Shukla et al.(1989).

With this in view, in chapter VIII of this thesis, a model is proposed to study the effect of changes in the habitat on growth

and survival of species that are caused by the depletion of forestry resources in their habitat due to cumulative effect of human and cattle populations in changing the carrying capacity of the habitat, Kormondy(1986). A conservation model is also proposed and analysed (see chapter VIII of this thesis for details).

1.2.2 OVER GROWTH OF A PARTICULAR SPECIES IN THE HABITAT

It is well known that one wild species in the habitat may grow uncontrolled affecting the growth and survival of other species. In the aquatic environment, a typical example of growth of noxious weeds (Eichhornia, Salvinia), which are harmful to other species, has been pointed out by many investigators, Thomas(1981), Webber(1978). Such changes in the habitat may adversely affect growth, migration and dispersion of certain other species, Whittaker(1967), Cody and Diamond(1975), Connell(1978), Luckinbill (1979), Kormondy(1986). Similar situation also exists in a large partly wetland Keoladeo National Park at Bharatpur(Rajasthan) in India. Here the habitat is degrading due to uncontrolled growth of wild grasses such as Paspalum distichum causing decrease in the population of phytoplancton, zooplankton, small and large fishes, birds, etc., Ali and Vijayan(1986). It has been pointed out by Ali and Vijayan(1986) on their ecological study on Keoladeo National Park that the wetland area of the park is being slowly and slowly converted into grassland - woodland biotype due to excessive growth of amphibious grass (Paspalum distichum). It has been recommended by Ali and Vijayan (1986) that the following feasible methods should be conducted to control the growth rate of Paspalum distichum.

(a) Bulldozing the area during dry season

(b) Burning grasses from selected areas

(c) Allowing buffaloes into the Park for grazing in limited numbers

It has been suggested that the cutting of the grasses by labour force or by allowing buffaloes to graze them would be more feasible.

Keeping this in view, in the last chapter of this thesis a mathematical model is proposed to study the changes in the habitat caused by the uncontrol growth of wild grasses such as Paspalum distichum on growth and existence of other wildlife species living in the same habitat (see chapter IX of this thesis for details).

1.3 MATHEMATICAL TECHNIQUES USED IN THE THESIS

In the deterministic analysis of evolution and stability of the systems described above many mathematical approaches have been adopted. In the present thesis only the following two approaches have been adopted.

1.3.1 THE METHOD OF CHARACTERISTIC ROOTS

The conclusions regarding asymptotic stability of the systems very much lie in the eigen values of the variational matrix, a Jacobian matrix of first order derivatives of interaction-functions. As this Jacobian is determined by Taylor expansion of the interaction-functions and neglecting nonlinear higher order terms, this method studies only the local stability of the system in vicinity of its equilibrium state. Being a straight forward method, based purely on the signs of real parts of the eigen values, Routh-Hurwitz criterion (Sanchetz(1968) and Gershgorin's theorem (Lancaster and Tismanetsky(1985)) are very useful to study the local stability of wide range of systems in homogeneous environments.

1.3.2 LIAPUNOV'S DIRECT METHOD

The physical validity of this method is contained in the fact that stability of the system depend on the energy of the system which is a function of system variables. Liapunov's direct method consists in finding out such energy functions termed as Liapunov functions which need not be unique. The major role in this process is played by positive or negative definite functions which can be obtained in general by trial of some particular functions of state variables, and in some cases with a planned procedure. The two basic theorems on stability can be found in La Salle and Lefschetz(1961).

The stability analysis in conjunction with a suitable Liapunov function has its two salient features. First-one is its straight forwardness being a direct method. Its second feature is that this procedure provides a realistic study of the stability of even multispecies systems.

1.4 SUMMARY OF THE THESIS

The thesis consists of nine chapters.

In chapter I, a general introduction with relevant literature is presented to provide a necessary back ground required for the forthcoming chapters.

In chapter II, a dynamic model for the depletion of forestry resource due to industrialization and pollution is proposed and analysed. It is assumed that the pollutant in the environment is exhausted with a prescribed rate (i. e. instantaneous, constant, industrialization dependent or periodic) and is depleted by some natural degradation factors. It is considered that the growth rate of resource biomass density decreases as the uptake concentration of pollutant increases while its carrying capacity decreases with

the concentration of pollutant present in the environment as well as with the density of industrialization. It is considered that the density of resource biomass is governed by generalized logistic equation. It is further considered that the density of industrialization is wholly dependent on the density of resource and it is governed by prey-predator type equation. It is assumed that the uptake rate of pollutant by the resource biomass is proportional to the density of resource and the concentration of pollutant present in the environment. By analysing the model it is shown that under instantaneous introduction of pollutant in the environment the resource may recover back to a level whose magnitude will depend upon the equilibrium state of industrialization. However, if the pollutant is continued to be exhausted in the environment with a constant rate, industrialization dependent rate or periodic rate, the forestry resource will eventually become extinct faster with industrialization than without it.

A model to conserve the resource biomass by reforestation and by controlling industrialization and pollution is also proposed and analysed. It is shown that an appropriate level of resource biomass can be maintained with the use of efforts required for the control of undesired level of industrialization and pollution.

In chapter III, a mathematical model to study the depletion of forestry resource due to combined effect of industrialization and population is proposed and analysed. It is assumed that the growth rate of resource biomass as well as its carrying capacity in the habitat decrease with the increase in densities of industrialization and population. It is also assumed that the density of industrialization wholly depends upon the resource and

its dynamics is governed by predation type process. By analysing the model it is shown that if the pressures due to industrialization and population continue without control, the forestry resource biomass may not last long leading to undesirable ecological and environmental consequences. However, by incorporating a conservation model it is shown that if suitable measures to control industrialization and population are taken, an appropriate level of resource biomass can be maintained.

In the chapter IV, a mathematical model is proposed to study the depletion of forestry resource in a habitat due to increase in the densities of population and pollution. It is assumed that the resource biomass and population densities are governed by logistic equations. It is further assumed that the pollutant is emitted into the environment with instantaneous, constant, population dependent or periodic rate. It is considered that the growth rate of resource density decreases with the density of population as well as with the uptake concentration of pollutant by the resource biomass. It is further considered that the carrying capacity of the resource decreases as the density of population and environmental concentration of pollutant increase. By analysing the model, it is shown that in the case of instantaneous spill of the pollutant in the environment the forestry resource biomass density will settle down to a lower equilibrium than its carrying capacity, the magnitude being dependent on the equilibrium level of the population density. However, if the pollutant is emitted in the environment with constant, population dependent or periodic rate, the equilibrium level of the density of resource biomass depends not only on the equilibrium level of population but also on the concentration of toxicant in the environment and is less

than its carrying capacity. A conservation model is also proposed, the analysis of which shows that the resource biomass can be maintained at an appropriate level by conserving the biomass and by controlling the population and pollution.

In the chapter V, a mathematical model is proposed to study the simultaneous effects of two toxicants, one is more toxic than the other, on the growth of a renewable resource such as a forest biomass. The cases of instantaneous spill, constant and periodic emission of each of the two toxicants in the environment are considered. It is assumed that the growth rate of the density of resource decreases as the uptake concentration of either of the two toxicants increases while the carrying capacity decreases due to increase in the environmental concentration of the two toxicants. By analysing the model, it is shown that if the toxicants are emitted with constant rates, the resource biomass is doomed to extinction sooner than the case of a single toxicant.

A model to conserve the density of resource biomass by reforestation and by controlling the toxicants is also proposed and analysed. It is shown that if suitable efforts are adopted to conserve the biomass and to control the toxicants, an appropriate level of the resource biomass can be maintained.

In chapter VI, a mathematical model is proposed and analysed to investigate the effects of two environmental factors such as acid rain and wind erosion on the depletion of fertile topsoil depth. It is assumed that the natural growth rate of fertile topsoil depth is either zero or a positive constant. It is further assumed that the growth rate of fertile topsoil decreases as the concentration of acid rain (deposited with a constant rate) increases or as the wind velocity increases (the velocity being

dependent upon pressure gradient which may be a constant or periodic). When the growth rate of topsoil depth is zero, it is shown that as the concentration of acid rain or the magnitude of the pressure gradient of wind increases the depth of fertile topsoil always tends to zero. However, when the natural growth rate of topsoil depth is a constant, it is shown that the increase in the concentration of acid rain or wind pressure gradient will lower the depth of fertile topsoil and it may tend to zero even in this case also if these factors remain uncontrolled. When the pressure gradient of wind is periodic with small amplitude, it is shown that a periodic behavior occurs in the system and its stability behavior is same as that of the case of constant pressure gradient.

A model to conserve the topsoil depth and to control acid rain and wind is also proposed and analysed. It is assumed here that the density of the effort applied to conserve the topsoil depth is proportional to the depleted depth of topsoil and the densities of the efforts applied to control the concentration of acid rain and the wind velocity by growing hedges are proportional to their respective undesired levels. By analysing the model it is shown that an appropriate level of fertile topsoil depth can be maintained.

In chapter VII, a mathematical model to study the depletion of forestry resource by two interacting populations is proposed and analysed. Four types of interactions between two populations (two kinds of industrialization) are taken into account i.e. (i) two competing populations (ii) one population is an alternative resource for the other, (iii) one population is a prey type resource for the other predator type population and (iv) two

populations are cooperating each other. In each case local and global stability behavior of the system are determined. In particular, it is shown that the effects of populations are to decrease the density of the resource biomass and the resource depletion is maximum when the two populations are cooperating and it is minimum when the two populations are competing in the corresponding cases of forestry resource population interactions.

In chapter VIII, a mathematical model is proposed and analysed to study the growth and survival of resource biomass dependent wild life species in a forested habitat which is being depleted due to cumulative effect of human and cattle populations. It is assumed that the carrying capacity of the habitat decreases due to cumulative growth of human and cattle populations and related developmental activities. It is further assumed that the growth rate of wild life species increases with the density of resource biomass. It is shown that as the cumulative density of human and cattle populations increases the resource density decreases leading to lowering of the density of wild life species and its eventual extinction if the population pressure continues unabatedly. A conservation model for resource biomass by reforestation and by controlling the population density is also proposed and analysed. It is shown that by conserving the resource biomass and by controlling the undesired level of cumulative density of human and cattle populations, an appropriate level of resource density can be maintained and the survival of resource dependent wild life species can be ensured.

In chapter IX, a mathematical model is proposed and analysed to study the effect of ecological changes caused by the growth of wild grasses such as Paspalum distichum over a large part of the

wetland wildlife park in Bharatpur, India, known as Keoladeo National Park. In the model the growth rate of cumulative density of flora and fauna such as phytoplancton, zooplankton, fishes, birds, etc. and the carrying capacity of the habitat are assumed to decrease with the increase in cumulative biomass density of the wild grasses. By analysing the model it is shown that if the wild grasses are not controlled the survival of the various species mentioned above will be threatened. It has been suggested by the model study that if the growth of wild grasses is controlled by allowing buffaloes to graze them or by cutting them, then the flora and fauna in the habitat will boom. The model presented here is applicable to other wetland parks around the world having similar ecology.

It is hoped that the investigations prescribed in this thesis will provide a basis for further study of a very important problem of resource depletion and its conservation needed for socio-economic sustainable development.

CHAPTER II

MODELLING THE DEPLETION AND CONSERVATION OF FORESTRY RESOURCES: EFFECTS OF INDUSTRIALIZATION AND POLLUTION

2.0 INTRODUCTION

It is well known in ecological studies that ecosystems are affected by various pollutants(toxicants) such as dissolved petroleum products, chemical toxicants, detergents, heavy metals present in the environment, Nelson(1970), Hass(1981), Jenson and Marshall(1982), Patin(1982). The presence of pollutants in the environment may decrease the growth rate of species and may also force them to migrate from the habitat leading to decrease in the quantity as well as quality of the yield, Clark(1976).

In recent decades, some investigators have studied the effect of pollutants on various ecosystem by utilizing mathematical models, Hallam and Clark (1982), Hallam et al. (1983), Hallam and De Luna (1984), De Luna and Hallam (1987). In particular, Hallam et al.(1983) have modelled the effect of toxicant in the environment on single species population by assuming that its growth rate density decreases linearly with the concentration of toxicant but the corresponding carrying capacity does not depend upon the concentration of the toxicant present in the environment. However, in real situation the effect of toxicant is to decrease both the growth rate of species as well as the carrying capacity of the environment. Taking this aspect into account Freedman and Shukla(1991) proposed and analysed a model to study the effect of toxicant on single species and predator prey system by taking into account of exogenous introduction of toxicant in the environment.

But in the model they did not consider the effect of industrialization which is the main cause of emission of toxicant in the environment.

In this chapter we, therefore, consider a dynamical model to study the effects of toxicant emitted by industries on biological species such as plant/tree population in a forest stand. It is assumed that the pollutant in the environment is exhausted with a rate which is dependent on the density of industrialization and is depleted by some natural degradation factors. The density of industrialization is assumed to be wholly dependent upon the resource such as in predation process. It is considered that the growth rate of concentration of pollutant uptaken by the species which causes decrease in its growth rate is proportional to the density of species population and the environmental concentration of the pollutant. It is further considered that the carrying capacity of the environment with respect to the species decreases with increase in the density of of industrialization as well as the concentration of the pollutant in the environment. The analysis of the model is carried out for four cases: (i) with instantaneous introduction, (ii) with constant rate of introduction, (iii) industrialization dependent introduction and (iv) periodic introduction of pollutant in the environment. Stability theory of ordinary differential equation is used for the model analysis.

2.1 MATHEMATICAL MODEL

As discussed above we consider a biological species such as plant/tree population in a forest stand (i.e. forestry resource biomass) affected by the pollutant emitted into the environment by

different types of industrial processes. We assume that the growth rate of species decreases by the uptake of pollutant by the species and the corresponding carrying capacity decreases with the increase in the density of industrialization as well as the density of pollutant in the environment. The density of industrialization is assumed to be wholly dependent upon the resource species and the interaction is prey-predator type. Thus this system can be written in the form of following differential equations (see Freedman and Shukla(1991)).

$$\frac{dB}{dt} = r_B(U)B - \frac{r_{BO}B^2}{K_B(I,T)} - \alpha BI$$

$$\frac{dI}{dt} = - \gamma_0 I - \gamma_1 I^2 + \beta IB$$

$$\frac{dT}{dt} = Q - \delta_0 T - \alpha_1 BT + \pi\nu BU$$

$$\frac{dU}{dt} = - \delta_1 U + \alpha_1 BT - \nu BU$$

(2.1)

$B(0) \geq 0$, $I(0) \geq 0$, $T(0) \geq 0$, $U(0) \geq c_0 B(0)$ and $0 \leq \pi \leq 1$.

Here $B(t)$ is the density of plant population in a forest stand (forestry resource biomass), $I(t)$ is the density of industrialization, $T(t)$ and $U(t)$ are the concentrations of pollutants in the environment and and in the plant population, respectively for any $t \geq 0$. Further γ_0 is the natural depletion rate coefficient and γ_1 is the intraspecific coefficient of industrialization, β is growth rate coefficient of industrialization depending wholly on the resource biomass. Also α is the depletion rate coefficient of the resource biomass due to industrialization, δ_0 and δ_1 are the depletion rates of toxicants in the environment and species biomass respectively, α_1 is the rate of depletion of pollutant in the environment due to uptake of

pollutant by the species population. Also some population may die out at a rate ν due to pollutant and a fraction π of this may again reenter in the environment.

Q is the rate of introduction of pollutant into the environment which is of four types as described below :

Q = 0, Q = Q_0 (a positive constant),

Q = Q(I) satisfying

$$Q(0) > 0, \ Q'(I) > 0 \tag{2.2a}$$

and Q is a periodic function i. e

$$Q = Q(t), \ Q(t) = Q_0 + \varepsilon \ \phi(t), \ \phi(t+\omega) = \phi(t) \tag{2.2b}$$

In our model (2.1), the function $r_B(U)$ represents the growth rate constant of consumer species which decreases with U and hence

$$r_B(0) = r_{B0} > 0 \ , \ \frac{\partial r_B(U)}{\partial U} < 0 \quad \text{for } U \geq 0$$

and $\quad r_B(\bar{\bar{U}}) = 0 \quad$ for some $\quad \bar{\bar{U}} > 0.$ $\tag{2.3}$

The function $K_B(I,T)$ denotes the maximum density of the resource biomass which the environment can support. It is assumed to be decreasing function of I and T and hence we have

$$K_B(0,0) = K_{B0} > 0, \ \frac{\partial K_B(I,T)}{\partial I} < 0, \ \frac{\partial K_B(I,T)}{\partial T} < 0 \text{ for } I \geq 0, \ T \geq 0$$

$$\tag{2.4}$$

i.e. for sufficiently high level of pollutant, biomass can not grow and in fact will die.

Further in the condition $U(0) \geq c_0 B(0)$, $c_0 \geq 0$ is the proportionality constant determining the measure of initial pollutant concentration in the biomass at t = 0.

2.2 MATHEMATICAL ANALYSIS

CASE I: $Q = 0$ (INSTANTANEOUS INTRODUCTION)

In this case, our model (2.1) has three nonnegative equilibria in B-I-T-U space, namely $E_0(0,0,0,0)$, $E_1(K_{BO},0,0,0)$, and $E_2(\tilde{B}_1,\tilde{I}_1,0,0)$. Here $\tilde{B}_1$ and $\tilde{I}_1$ are the positive solution of

$$\alpha I = r_{BO}\left(1 - \frac{B}{K_B(I,0)}\right) \tag{2.5a}$$

$$\gamma_1 I = \beta B - \gamma_0 \tag{2.5b}$$

It is easy to check that the isoclines (2.5a) and (2.5b) intersect at a unique point $(\tilde{B}_1,\tilde{I}_1)$ iff

$$K_{BO} > \gamma_0/\beta \tag{2.6}$$

Let M_i be the variational matrices corresponding to E_i, $i = 0, 1, 2$. Then we have

$$M_0 = \begin{bmatrix} r_{BO} & 0 & 0 & 0 \\ 0 & -\gamma_0 & 0 & 0 \\ 0 & 0 & -\delta_0 & 0 \\ 0 & 0 & 0 & -\delta_1 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} -r_{BO} & r_{BO}\dfrac{\partial K_B(0,0)}{\partial I} - \alpha K_{BO} & r_{BO}\dfrac{\partial K_B(0,0)}{\partial T} & K_{BO}\dfrac{\partial r_B(0)}{\partial U} \\ 0 & \beta K_{BO} - \gamma_0 & 0 & 0 \\ 0 & 0 & -\delta_0 - \alpha_1 K_{BO} & \pi\nu K_{BO} \\ 0 & 0 & \alpha_1 K_{BO} & -\delta_1 - \nu K_{BO} \end{bmatrix}$$

$$
M_2 = \begin{bmatrix}
-\dfrac{r_{BO}\tilde{B}_1}{K_B(\tilde{I}_1,0)} & F_{12} & F_{13} & \tilde{B}_1\,\dfrac{\partial r_B(0)}{\partial U} \\[4mm]
\beta\tilde{I}_1 & -\gamma_1\tilde{I}_1 & 0 & 0 \\[4mm]
0 & 0 & -\delta_0 - \alpha_1\tilde{B}_1 & \pi\nu\tilde{B}_1 \\[4mm]
0 & 0 & \alpha_1\tilde{B}_1 & -\delta_1 - \nu\tilde{B}_1
\end{bmatrix}
$$

where

$$
F_{12} = \frac{r_{BO}\tilde{B}_1^2}{K_B^2(\tilde{I}_1,0)}\,\frac{\partial K_B(\tilde{I}_1,0)}{\partial I} - \alpha\tilde{B}_1 \tag{2.7a}
$$

$$
F_{13} = \frac{r_{BO}\tilde{B}_1^2}{K_B^2(\tilde{I}_1,0)}\,\frac{\partial K_B(\tilde{I}_1,0)}{\partial T} \tag{2.7b}
$$

From the above variational matrices we note that E_0 is a saddle point with stable manifold locally in I-T-U space and unstable manifold locally in B direction. E_1 is also a saddle point with stable manifold locally in B-T-U space and unstable manifold locally in I direction ($\beta K_{BO} - \gamma_0$ is taken positive). Using Routh-Hurwitz criterion we note from M_2 that E_2 is locally asymptotically stable.

In the following theorem we are able to show that E_2 is globally asymptotically stable.

THEOREM 2.2.1   If $B(0) > 0$, $I(0) > 0$, then E_2 is globally asymptotically stable.

Proof: From (2.1) we have

$$\frac{dB}{dt} = r_B(U)B - \frac{r_{B0}B^2}{K_B(I,T)} - \alpha BI$$

$$\leq r_0 B - r_0 B^2/K_{B0}$$

and hence $\lim_{t\to\infty} B(t) \leq K_{B0}$

Again we have

$$\frac{dI}{di} = -\gamma_0 I - \gamma_1 I^2 + \beta IB$$

$$\leq (\beta K_{B0} - \gamma_0)I - \gamma_1 I^2$$

This implies that $\lim_{t\to\infty} I(t) \leq (\beta K_{B0} - \gamma_0)/\gamma_1$

We also have

$$\frac{dT}{dt} + \frac{dU}{dt} = -\delta_0 T - \delta_1 U - (1 - \pi)\nu BU$$

$$\leq -\delta(T+U), \quad \text{where} \quad \delta = \min(\delta_0, \delta_1)$$

and hence $T(t)+U(t) \leq \left[T(0)+U(0)\right] e^{-\delta t}$.

This implies $\lim_{t\to\infty} T(t) = \lim_{t\to\infty} U(t) = 0$

This shows that the system is dissipative. Since $B(0) > 0$, $I(0) > 0$, the theorem follows.

The above theorem shows that under instantaneous introduction of pollutant in the environment the species can return back to a level which is lower than its carrying capacity, the magnitude of which would depend upon the equilibrium level of density of industrialization.

CASE II: $Q = Q_0 > 0$ (CONSTANT EMISSION RATE)

In this case the model (2.1) has three nonnegative equilibria, namely $E_1^*(0,0,Q_0/\delta_0,0)$, $E_2^*(\tilde{B},0,\tilde{T},\tilde{U})$ and $E^*(B^*,I^*,T^*,U^*)$. The existence of E_1^* is obvious. Now we shall show

the existence of other two equilibria as follows.

Existence of $E_2^*(\tilde{B}, 0, \tilde{T}, \tilde{U})$:

Here $\tilde{B}$, $\tilde{T}$ and $\tilde{U}$ are the positive solution of the system of algebraic equations

$$B = \frac{1}{r_{B0}} r_B(U) K_B(0, g(B)) \tag{2.8a}$$

$$T = g(B) \tag{2.8b}$$

$$U = h(B) \tag{2.8c}$$

where

$$g(B) = \frac{Q_0 + \pi\nu B h(B)}{\delta_0 + \alpha_1 B} \tag{2.9a}$$

$$h(B) = \frac{\alpha_1 Q_0 B}{f(B)} \tag{2.9b}$$

$$f(B) = \delta_0 \delta_1 + (\delta_0 \nu + \delta_1 \alpha_1) B + (1 - \pi)\nu\alpha_1 B^2 \tag{2.9c}$$

It can be easily checked that in the isocline (2.8a), U is a decreasing function of B starting from $\bar{\bar{U}}$ iff

$$r_B(U) \frac{\partial K_B}{\partial g} \frac{dU}{dB} < r_{B0} \tag{2.10}$$

Also in the isocline (2.8c), U is an increasing function of B starting from zero iff

$$\delta_0 \delta_1 > (1 - \pi)\nu\alpha_1 B^2 \tag{2.11}$$

Thus the two isoclines (2.8a) and (2.8c) must intersect at a unique point under the conditions (2.10) and (2.11). The intersection value of the above isoclines gives B-U coordinates of E_2^* and its T-coordinate can be determined from (2.8b).

Existence of $E^*(B^*, I^*, T^*, U^*)$:

Here B^*, I^*, T^* and U^* are the positive solution of the system of algebraic equations

$$\alpha I = r_B(h^*(B)) - \frac{r_{B0}B}{K_B(I,g^*(B))} \tag{2.12a}$$

$$\gamma_1 I = \beta B - \gamma_0 \tag{2.12b}$$

$$T = g^*(B) \tag{2.12c}$$

$$U = h^*(B) \tag{2.12d}$$

where

$$g^*(B) = \frac{Q_0 + \pi\nu B h^*(B)}{\delta_0 + \alpha_1 B} \tag{2.13a}$$

$$h^*(B) = \frac{\alpha_1 Q_0 B}{f^*(B)} \tag{2.13b}$$

$$f^*(B) = \delta_0\delta_1 + (\delta_0\nu + \delta_1\alpha_1)B + (1 - \pi)\nu\alpha_1 B^2 \tag{2.13c}$$

From (2.8a) and (2.12a) we note that in (2.12a) the equilibrium level of species is lower than the equilibrium level of species in (2.8a). This shows that if industrialization and toxicant both act on the species, then the decrease in the density of species will be more than the case when only either industrialization or pollution is taken into account.

From (2.12a) we note the following

$$\text{When } B \rightarrow 0, \quad I \rightarrow r_{B0}/\alpha \tag{2.14a}$$

$$\text{When } I \rightarrow 0, \quad B \rightarrow B_a \tag{2.14b}$$

where B_a is a zero of

$$F(B) = r_{B0}B - r_B(h^*(B))K_B(0,g^*(B)) \tag{2.14c}$$

We note that $F(0) < 0$ and $F(K_{B0}) > 0$ and hence there exists B_a in the interval $0 < B_a < K_{B0}$ such that $F(B_a) = 0$.

We can also check that $\frac{dI}{dB}$ computed from (2.12a) is negative iff

$$\frac{\partial r_B}{\partial U}\frac{dh^*}{dB} - \frac{r_{BO}B}{K_B(I,g^*(B))} + \frac{r_{BO}B}{K_B^2(I,g^*(B))}\frac{\partial K_B}{\partial T}\frac{dg^*}{dB} < 0 \qquad (2.15a)$$

Further it is clear that the isocline (2.12b) represents the equation of a straight line passing through $(0, -\gamma_0/\gamma_1)$ and $(\gamma_0/\beta, 0)$.

Thus the isoclines (2.12a) and (2.12b) will intersect at a unique point [see fig. 2.1], provided

$$\gamma_0/\beta < B_a < K_{BO} \qquad (2.15b)$$

The intersection value of these two isoclines gives the B^* and I^* coordinates of E^*. Finally T^* and U^* can be computed from (2.12c) and (2.12d) respectively. This completes the existence of E^*.

To study the local stability behavior of the equilibria, we compute the variational matrices corresponding to each equilibrium as follows.

$$M_1^* = \begin{bmatrix} r_{BO} & 0 & 0 & 0 \\ 0 & -\gamma_0 & 0 & 0 \\ -\dfrac{\alpha_1 Q_0}{\delta_0} & 0 & -\delta_0 & 0 \\ \dfrac{\alpha_1 Q_0}{\delta_0} & 0 & 0 & -\delta_1 \end{bmatrix}$$

$$M_2^* = \begin{bmatrix} -r_B(\tilde{U}) & G_{12} & G_{13} & \tilde{B}\dfrac{\partial r_B(\tilde{U})}{\partial U} \\[2mm] 0 & \beta\tilde{B} - \gamma_0 & 0 & 0 \\[2mm] -\alpha_1\tilde{T} + \pi\nu\tilde{U} & 0 & -\delta_0 - \alpha_1\tilde{B} & \pi\nu\tilde{B} \\[2mm] \alpha_1\tilde{T} - \nu\tilde{U} & 0 & \alpha_1\tilde{B} & -\delta_1 - \nu\tilde{B} \end{bmatrix}$$

Where

$$G_{12} = \frac{r_{BO}\tilde{B}^2}{K_B^2(\tilde{I},0)}\frac{\partial K_B(\tilde{I},0)}{\partial I} - \alpha\tilde{B} \tag{2.16a}$$

$$G_{13} = \frac{r_{BO}\tilde{B}^2}{K_B^2(\tilde{I},0)}\frac{\partial K_B(\tilde{I},0)}{\partial T} \tag{2.16b}$$

$$M^* = \begin{bmatrix} -\dfrac{r_{BO}B^*}{K_B(I^*,T^*)} & H_{12} & H_{13} & B^*\dfrac{\partial r_B(U^*)}{\partial U} \\[2mm] \beta I^* & -\gamma_1 I^* & 0 & 0 \\[2mm] -\alpha_1 T^* + \pi\nu U^* & 0 & -\delta_0 - \alpha_1 B^* & \pi\nu B^* \\[2mm] \alpha_1 T^* - \nu U^* & 0 & \alpha_1 B^* & -\delta_1 - \nu B^* \end{bmatrix}$$

where

$$H_{12} = \frac{r_{BO}B^{*2}}{K_B^2(I^*,T^*)}\frac{\partial K_B(I^*,T^*)}{\partial I} - \alpha B^* \tag{2.17a}$$

$$H_{13} = \frac{r_{BO}B^{*2}}{K_B^2(I^*,T^*)}\frac{\partial K_B(I^*,T^*)}{\partial T} \tag{2.17b}$$

From M_1^* we note that E_1^* is one dimensional unstable and three dimensional stable manifold. From M_2^* we note that E_2^* is locally

unstable in I direction and its stability behavior in B-T-U space is discussed in Freedman and Shukla(1991).

In the following theorem we have shown that E^* is locally asymptotically stable under certain conditions.

THEOREM 2.2.2 Let the following inequalities hold:

$$\frac{r_{BO}B^*}{K_B(I^*,T^*)} > \beta I^* + 2\alpha_1 T^* - (1 + \pi)\nu U^* \tag{2.18a}$$

$$\gamma_1 I^* > - H_{12} \tag{2.18b}$$

$$\delta_0 > - H_{13} \tag{2.18c}$$

$$\delta_1 + (1 - \pi)\nu B^* > - B^* \frac{\partial r_B(U^*)}{\partial U} \tag{2.18d}$$

Proof: If the inequalities (2.18) hold, then by Gershgorin's theorem (Lancaster and Tismanetsky, p.371, 1985) all eigen values of M^* will have negative real parts, and the theorem follows.

In the following theorem we have shown that E^* is globally asymptotically stable under certain conditions. To prove this theorem we first require the following lemma which establishes the region of attraction for our system.

LEMMA 2.2.1 The set
$$A = \left\{(B,I,T,U) = 0 \leq B \leq K_{BO}, \ 0 \leq I \leq I_m, \ 0 \leq T + U \leq Q_0/\delta\right\} \text{ is a}$$
region of attraction for all the solutions initiating in the positive octant, where $I_m = (\beta K_{BO} - \gamma_0)/\gamma_1$, $\delta = \min(\delta_0, \delta_1)$.

Proof: As before, $\lim_{t\to\infty} B(t) \leq K_{BO}$

$$\lim_{t\to\infty} I(t) \leq I_m$$

and $\frac{dT}{dt} + \frac{dU}{dt} \leq Q_0 - \delta(T+U)$

Hence $\lim_{t\to\infty} [T(t) + U(t)] \leq Q_0/\delta$, proving the lemma.

THEOREM 2.2.3   In addition to the assumptions (2.3) and (2.4), let $r_B(U)$ and $K_B(I,T)$ satisfy in A,

$$K_m \leq K_B(I,T) \leq K_{BO}, \quad 0 \leq -\frac{\partial r_B(U)}{\partial U} \leq \rho,$$

$$0 \leq -\frac{\partial K_B(I,T)}{\partial I} \leq k_1, \quad 0 \leq -\frac{\partial K_B(I,T)}{\partial T} \leq k_2, \quad (2.19)$$

for some positive constants K_m, ρ, k_1, k_2.

Then if the following inequalities hold

$$\left[\frac{r_{BO} K_{BO} k_1}{K_m^2} + \alpha + \beta \right]^2 < \frac{4}{3} \frac{r_{BO} \gamma_1}{K_B(I^*,T^*)} \quad (2.20a)$$

$$\left[\frac{r_{BO} K_{BO} k_2}{K_m^2} + \frac{\alpha_1 Q_0}{\delta} + \pi \nu U^* \right]^2 < \frac{2}{3} \frac{r_{BO}}{K_B(I^*,T^*)} (\delta_0 + \alpha_1 B^*) \quad (2.20b)$$

$$\left[\rho + \frac{\nu Q_0}{\delta} + \alpha_1 T^* \right]^2 < \frac{2}{3} \frac{r_{BO}}{K_B(I^*,T^*)} (\delta_1 + \nu B^*) \quad (2.20c)$$

$$\left[\pi \nu + \alpha_1 \right]^2 K_{BO}^2 < (\delta_0 + \alpha_1 B^*)(\delta_1 + \nu B^*) \quad (2.20d)$$

E^* is globally asymptotically stable with respect to all solutions initiating in the interior of the positive octant.

Proof: Consider the following positive definite function about E^*,

$$V(B,I,T,U) = [B - B^* - B^* \ln \frac{B}{B^*}] + [I - I^* - I \ln \frac{I}{I^*}]$$

$$+ \frac{1}{2} (T - T^*)^2 + \frac{1}{2} (U - U^*)^2 \quad (2.21)$$

Differentiating V with respect to t along the solution of (2.1), a little computation yields

$$\frac{dV}{dt} = -\frac{r_{B0}}{K_B(I^*,T^*)} (B - B^*)^2 - \gamma_1 (I - I^*)^2$$

$$- (\delta_0 + \alpha_1 B^*)(T - T^*)^2 - (\delta_1 + \nu B^*)(U - U^*)^2$$

$$+ (B - B^*)(I - I^*)\left[-r_{B0}B\, \xi_1(I,T) + \beta - \alpha \right]$$

$$+ (B - B^*)(T - T^*)\left[-r_{B0}B\, \xi_2(I^*,T) - \alpha_1 T + \pi\nu U^* \right]$$

$$+ (B - B^*)(U - U^*)\left[\eta(U) + \alpha_1 T^* - \nu U \right]$$

$$+ (T - T^*)(U - U^*)\left[\alpha_1 B + \pi\nu B \right] \tag{2.22}$$

where

$$\eta(U) = \begin{cases} \left[r_B(U) - r_B(U^*) \right]/(U - U^*) & U \neq U^* \\[2mm] \dfrac{\partial r_B(U^*)}{\partial U} & U = U^* \end{cases} \tag{2.23a}$$

$$\xi_1(I,T) = \begin{cases} \left[\dfrac{1}{K_B(I,T)} - \dfrac{1}{K_B(I^*,T)} \right]/(I - I^*), & I \neq I^* \\[4mm] -\dfrac{1}{K_B^2(I^*,T)}\; \dfrac{\partial K_B(I^*,T)}{\partial I}, & I = I^* \end{cases} \tag{2.23b}$$

$$\xi_2(I^*,T) = \begin{cases} \left[\dfrac{1}{K_B(I^*,T)} - \dfrac{1}{K_B(I^*,T^*)} \right]/(T - T^*), & T \neq T^* \\[4mm] -\dfrac{1}{K_B^2(I^*,T^*)}\; \dfrac{\partial K_B(I^*,T^*)}{\partial T} & T = T^* \end{cases} \tag{2.23c}$$

From (2.19) and mean value theorem we note that

$$|\eta(U)| \leq \rho, \quad |\xi_1(I,T)| \leq k_1/K_m^2, \quad |\xi_2(I^*,T)| \leq k_2/K_m^2. \tag{2.24}$$

The equation (2.22) can further be written as sum of the quadratics

$$\frac{dV}{dt} = -\frac{1}{2} a_{11}(B - B^*)^2 + a_{12}(B - B^*)(I - I^*) - \frac{1}{2} a_{22}(I - I^*)^2$$

$$- \frac{1}{2} a_{11}(B - B^*)^2 + a_{13}(B - B^*)(T - T^*) - \frac{1}{2} a_{33}(T - T^*)^2$$

$$- \frac{1}{2} a_{11}(B - B^*)^2 + a_{14}(B - B^*)(U - U^*) - \frac{1}{2} a_{44}(U - U^*)^2$$

$$- \frac{1}{2} a_{33}(T - T^*)^2 + a_{34}(T - T^*)(U - U^*) - \frac{1}{2} a_{44}(U - U^*)^2$$

$$(2.25)$$

where

$$a_{11} = \frac{2}{3} \frac{r_{BO}}{K_B(I^*, T^*)} \ , \ a_{22} = 2\gamma_1, \ a_{33} = \delta_0 + \alpha_1 B^*,$$

$$a_{44} = \delta_1 + \nu B^* \ , \ a_{12} = -r_{BO}B \ \xi_1(I, T) + \beta - \alpha,$$

$$a_{13} = -r_{BO}B \ \xi_2(I^*, T) - \alpha_1 T + \pi\nu U^*, \ a_{14} = \eta(U) + \alpha_1 T^* - \nu U,$$

$$a_{34} = \alpha_1 B + \pi\nu B.$$

Then sufficient conditions for $\frac{dV}{dt}$ to be negative definite are that

$$a_{12}^2 < a_{11}a_{22} \tag{2.26a}$$

$$a_{13}^2 < a_{11}a_{33} \tag{2.26b}$$

$$a_{14}^2 < a_{11}a_{44} \tag{2.26c}$$

$$a_{34}^2 < a_{33}a_{44} \tag{2.26d}$$

hold. We note that (2.20a,b,c,d) $\Rightarrow$ (2.26a,b,c,d) respectively. Hence V is a Liapunov function with respect to E^* whose domain contains the region A, proving the theorem.

This theorem shows that if the inequalities (2.20) hold, the system settles down to a steady state but at a lower equilibrium

level than the case without industrialization or toxicant and this level is determined by the uptake rate and washout rate of toxicant and the rate of density of industrialization. It should be noted that if the densities of industrialization and pollution increases without control, then the forest resource biomass may doom to extinction.

CASE III: $Q = Q(I)$ (INDUSTRIALIZATION DEPENDENT EMISSION)

In this case, the model (2.1) has again three nonnegative equilibria, namely $\hat{E}_1(0,0,Q(0)/\delta_0,0)$, $\hat{E}_2(\hat{B}_b,0,\hat{T}_b,\hat{U}_b)$ and $\hat{E}(\hat{B},\hat{I},\hat{T},\hat{U})$. The existence of $\hat{E}_2$ follows from the existence of E_2^* in case II and $\hat{B}_b$, $\hat{T}_b$, $\hat{U}_b$ can be obtained from (2.8) by replacing Q_0 by $Q(0)$. Also the existence of $\hat{E}$ can be seen along the same lines as that of E^* in case II and the value of $\hat{B}$, $\hat{I}$, $\hat{T}$, $\hat{U}$ can be computed from (2.12) by replacing Q_0 by $Q(f_1(B))$, where $f_1(B)$ is given by

$$f_1(B) = (\beta B - \gamma_0)/\gamma_1 \qquad (2.27)$$

Further by computing the variational matrices corresponding to $\hat{E}_1$ and $\hat{E}_2$, it can be seen that the local stability behavior of $\hat{E}_1$ and $\hat{E}_2$ is same as that of E_1^* and E_2^* of case II. The sufficient conditions for $\hat{E}$ to be locally asymptotically stable are given in the following theorem whose proof is similar to the theorem 2.2.2 and hence omitted.

THEOREM 2.2.4 Let the following inequalities hold

$$\frac{r_{B0}\hat{B}}{K_B(\hat{I},\hat{T})} > \beta\hat{I} + 2\alpha_1\hat{T} - (1 + \varkappa)\nu\hat{U} \qquad (2.28a)$$

$$\gamma_1\hat{I} > \alpha\hat{B} + Q'(\hat{I}) - \frac{r_{B0}\hat{B}^2}{K_B^2(\hat{I},\hat{T})}\frac{\partial K_B(\hat{I},\hat{T})}{\partial I} \qquad (2.28b)$$

$$\delta_0 \; > \; - \; \frac{r_{B0}\hat{B}^2}{K_B^2(\hat{I},\hat{T})} \; \frac{\partial K_B(\hat{I},\hat{T})}{\partial T} \tag{2.28c}$$

$$\delta_1 \; + \; (1 \; - \; \pi)\nu\hat{B} \; > \; - \; \hat{B} \; \frac{\partial r_B(\hat{U})}{\partial U} \tag{2.28d}$$

Then $\hat{E}$ is locally asymptotically stable.

In order to prove the global stability behavior of $\hat{E}$, we state the following lemma whose proof is similar to the lemma 2.2.1 and hence omitted.

LEMMA 2.2.2 The set
$$\hat{A} = \left\{ (B,I,T,U) \; : \; 0 \leq B \leq K_{B0}, \; 0 \leq I \leq I_m, \; 0 \leq T + U \leq Q_m/\delta \right\} \text{ is a}$$
region of attraction for all the solutions initiating in the positive octant, where $I_m = (\beta K_{B0} - \gamma_0)/\gamma_1$, $\delta = \min(\delta_0, \delta_1)$, and $Q_m = Q(I_m)$.

The proof of the following theorem is similar to the theorem 2.2.3 and hence omitted.

THEOREM 2.2.5 In addition to the assumptions (2.2a), (2.3) and (3.4), let $Q(I)$, $r_B(U)$ and $K_B(I,T)$ satisfy in $\hat{A}$,

$$\hat{K}_m \leq K_B(I,T) \leq K_{B0}, \quad 0 \leq - \frac{\partial r_B(U)}{\partial U} \leq \hat{\rho}, \quad 0 \leq Q'(I) \leq \hat{k},$$

$$0 \leq - \frac{\partial K_B(I,T)}{\partial I} \leq \hat{k}_1, \quad 0 \leq - \frac{\partial K_B(I,T)}{\partial T} \leq \hat{k}_2, \tag{2.29}$$

for some positive constants $\hat{K}_m$, $\hat{\rho}$, $\hat{k}$, $\hat{k}_1$, $\hat{k}_2$.

Then if the following inequalities hold

$$\left[\frac{r_{B0}K_{B0}\hat{k}_1}{\hat{K}_m^2} + \alpha + \beta \right]^2 < \frac{2}{3} \frac{r_{B0}\;\gamma_1}{K_B(\hat{I},\hat{T})} \tag{2.30a}$$

$$\left[\frac{r_{B0}K_{B0}\hat{k}_2}{\hat{K}_m^2} + \frac{\alpha_1 Q_m}{\delta} + \pi\nu\hat{U} \right]^2 < \frac{4}{9} \frac{r_{B0}}{K_B(\hat{I},\hat{T})} \; (\delta_0 + \alpha_1\hat{B}\;) \tag{2.30b}$$

$$\left[\hat{\rho} + \frac{\nu \, Q_m}{\delta} + \alpha_1 \hat{T} \right]^2 < \frac{2}{3} \frac{r_{BO}}{K_B(\hat{I}, \hat{T})} (\delta_1 + \nu \hat{B}) \qquad (2.30c)$$

$$\left[\pi \, \nu + \alpha_1 \right]^2 K_{BO}^2 < \frac{2}{3} (\delta_0 + \alpha_1 \hat{B})(\delta_1 + \nu \hat{B}) \qquad (2.30d)$$

$$\hat{k} < \frac{2}{3} \gamma_1 (\delta_0 + \alpha_1 \hat{B}) \qquad (2.30e)$$

$\hat{E}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive octant.

This theorem shows that if the inequalities (2.30) hold, then in the case of industrialization dependent emission of the pollutant $(Q = Q(I))$ in the environment, the system again settles down to a steady state whose magnitude would depend upon the influx and washout rates of the pollutant present in the environment, the influx rate being dependent upon the steady state industrialization.

CASE IV: $Q = Q(t)$, $Q(t) = Q_0 + \varepsilon \phi(t)$, $\phi(t+\omega) = \phi(t)$.

The system (2.1) can be written as

$$\dot{x} = A(x) + \varepsilon C(t), \quad x(0) = x_0 \qquad (2.31)$$

where $\cdot = \dfrac{d}{dt}$, $x = [x_1, x_2, x_3, x_4]^{tr} = [B, I, T, U]^{tr}$,

$$A(x) = \begin{bmatrix} r(x_4)x_1 - r_0 x_1^2/K(x_2, x_3) - \alpha x_1 x_2 \\ -\gamma_0 x_2 - \gamma_1 x_2^2 + \beta x_1 x_2 \\ Q_0 - \delta_0 x_3 - \alpha_1 x_1 x_3 + \pi \nu x_1 x_4 \\ -\delta_1 x_4 + \alpha_1 x_1 x_3 - \nu x_1 x_4 \end{bmatrix}$$

$C(t) = [0, 0, \varepsilon \phi(t), 0]^{tr}$, $\quad x_0 = [B(0), I(0), T(0), U(0)]^{tr}$,

tr = transpose.

Under an analysis similar to Freedman and Shukla (1991) we can establish the following two important results of this section.

THEOREM 2.2.6   If M^* has no eigen value with zero real parts, then the system (2.1) with $Q = Q(t) = Q_0 + \varepsilon\phi(t)$, $\phi(t+\omega) = \phi(t)$ has a periodic solution with period ω, $(B(t,\varepsilon),I(t,\varepsilon),T(t,\varepsilon),U(t,\varepsilon))$ such that $(B(t,0),I(t,0),T(t,0),U(t,0)) = (B^*,I^*,T^*,U^*)$.

THEOREM 2.2.7   If M^* has no eigen value with zero real parts, then for sufficiently small ε, the stability behavior of the periodic solution of the system (2.1) is same as that of E^*.

Moreover, the periodic solution up to order ε can be computed as

$$x(t,\xi,\varepsilon) = x^* + e^{M^*t} \left[\int_0^t e^{-M^*s} C(s)\, ds \right.$$
$$\left. - (e^{M^*\omega} - I)^{-1} e^{M^*\omega} \int_0^\omega e^{-M^*s} C(s)\, ds \right]\varepsilon + o(\varepsilon) \qquad (2.32)$$

where I is the identity matrix.

It is noted from the above theorems that a small periodic emission of the pollutant into the environment introduces the periodic behavior in the system but its stability characteristics remain the same as the constant case.

2.3   CONSERVATION MODEL

To conserve the resource biomass, which is being depleted due to continues pressure of industrialization and pollution, some kind of effort such as reforestation, irrigation, fencing, etc. has to be adopted.

Let $F_1(t)$, $F_2(t)$, $F_3(t)$ be the densities of efforts applied to increase the resource biomass $B(t)$ by reforestation, and to control the density of industrialization $I(t)$ and to control the concentration of toxicant $T(t)$ respectively. It is assumed that

$F_1(t)$ is proportional to the depleted level of resource biomass from its carrying capacity, $F_2(t)$ and $F_3(t)$ are proportional to the undesired level of the density of industrialization and concentration of pollutant in the environment. Then the dynamics of the system can be governed by the following system of differential equations:

$$\frac{dB}{dt} = r_B(U)B - \frac{r_{BO}B^2}{K_B(I,T)} - \alpha BI + r_{10}F_1$$

$$\frac{dI}{dt} = - \gamma_0 I - \gamma_1 I^2 + \beta IB - r_{20}F_2 I$$

$$\frac{dT}{dt} = Q - \delta_0 T - \alpha_1 BT + \pi\nu BU - r_{30}F_3$$

$$\frac{dU}{dt} = - \delta_1 U + \alpha_1 BT - \nu BU \qquad (2.33)$$

$$\frac{dF_1}{dt} = r_1(1 - B/K_{BO}) - \nu_1 F_1$$

$$\frac{dF_2}{dt} = r_2(I - I_c) - \nu_2 F_2$$

$$\frac{dF_2}{dt} = r_2(T - T_c) - \nu_3 F_3$$

$B(0) \geq 0$, $I(0) \geq 0$, $T(0) \geq 0$, $U(0) \geq c_0 B(0)$, $F_i(0) \geq 0$, $i = 1,2,3$ and $0 \leq \pi \leq 1$.

Here r_i denotes the constant growth rate coefficient of the effort F_i and ν_i its depreciation rate coefficient, $i = 1,2,3$. Further r_{10} is the growth rate coefficient of the resource biomass $B(t)$ due to the effort F_1, r_{20} and r_{30} are the depletion rate coefficients of $I(t)$ and $T(t)$ due to the efforts F_2 and F_3 respectively. I_c is the critical value of industrialization which is harmless to the resource biomass and T_c is the concentration of pollutant permissible and harmless to the biomass. Other notations

in the model (2.33) have the same meaning as in the model (2.1). In this case we analyse our model (2.33) for two cases when $Q = Q_0$ is a positive constant and $Q = Q(I)$ satisfying (2.2a).

CASE I: $Q = Q_0 > 0$

In this case there is only one interior equilibrium $\bar{E}(\bar{B}, \bar{I}, \bar{T}, \bar{U}, \bar{F}_1, \bar{F}_2, \bar{F}_3)$ of the model (2.33) and it is the positive solution of the system of the algebraic equations :

$$I = \frac{1}{\alpha} \left[r_B(\bar{h}(B)) - \frac{r_{B0}B}{K_B(I, \bar{g}(B))} + \frac{r_1 r_{10}}{\nu_1 B} (1 - \frac{B}{K_{B0}}) \right] \qquad (2.34a)$$

$$I = \frac{\beta\nu_2 B + r_2 r_{20} I_c - \nu_2 \gamma_0}{\gamma_1 \nu_2 + r_2 r_{20}} \qquad (2.34b)$$

$$T = \bar{g}(B), \quad U = \bar{h}(B), \quad F_1 = r_1(1 - B/K_{B0})/\nu_1,$$

$$F_2 = r_2(I - I_c)/\nu_2, \quad F_3 = r_3(T - T_c)/\nu_3 \qquad (2.34c)$$

where

$$\bar{h}(B) = \alpha_1 Q_0^* B / \bar{f}(B),$$

$$\bar{f}(B) = \delta_0^* \delta_1 + (\delta_0^* \nu + \delta_1 \alpha_1)B + (1 - \pi)\nu\alpha_1 B^2,$$

$$\bar{g}(B) = [Q_0^* + \pi\nu B \, \bar{h}(B)]/(\delta_0^* + \alpha_1 B),$$

$$Q_0^* = Q_0 + r_3 r_{30} T_c/\nu_3, \quad \delta_0^* = \delta_0 + r_3 r_{30}/\nu_3.$$

For the existence of $\bar{E}$, it suffices to show that the isoclines (2.34a) and (2.34b) intersect in the positive orthant. It can be checked that the isoclines (2.34a) and (2.34b) will intersect at a unique point [see fig. 2.2] iff

$$\frac{\partial r_B}{\partial U} \frac{d\bar{h}}{dB} - \frac{r_{B0}B}{K_B(I, \bar{g}(B))} + \frac{r_{B0}B}{K_B^2(I, \bar{g}(B))} \frac{\partial K_B}{\partial T} \frac{d\bar{g}}{dB} - \frac{r_1 r_{10}}{\nu_1 B^2} < 0 \qquad (2.35a)$$

and $r_2 r_{20} I_c > \nu_2 \gamma_0 \qquad (2.35b)$

REMARK: It should be pointed out here that even if the inequality in (2.35b) is reversed, the isoclines (2.34a) and (2.34b) will intersect at a unique point [see fig. 2.2] if in addition to (2.35a), the following inequality holds:

$$B_k < B_s \qquad (2.35c)$$

where $B_k = [\nu_2 \gamma_0 - r_2 r_{20} I_c]/\beta \nu_2$

and B_s is a solution in the interval $0 < B_s < K_{BO}$ of the following equation:

$$r_{BO}B = K_B(0, \bar{g}(B)\left[r_B(\bar{h}(B)) + \frac{r_1 r_{10}}{\nu_1 B} \left(1 - \frac{B}{K_{BO}}\right)\right]$$

The intersection value of the above two isoclines gives the $B - I$ coordinates of $\bar{E}$ and its other coordinates can be computed from (2.34c)

By computing the variational matrix corresponding the equilibrium $\bar{E}$, we can prove the following theorem whose proof is similar to the theorem 2.2.2 and hence we omit it.

THEOREM 2.3.1 Let the following conditions are satisfied :

$$\frac{r_{BO}\bar{B}}{K_B(\bar{I}, \bar{T})} + \frac{r_{10}\bar{F}_1}{\bar{B}} > \beta\bar{I} + 2\alpha_1\bar{T} - (1 + \pi)\nu\bar{U} + \frac{r_1}{K_{BO}} \qquad (2.36a)$$

$$\gamma_1\bar{I} > \alpha\bar{B} - \frac{r_{BO}\bar{B}^2}{K_B^2(\bar{I}, \bar{T})} \frac{\partial K_B(\bar{I}, \bar{T})}{\partial I} + r_2 \qquad (2.36b)$$

$$\delta_0 > - \frac{r_{BO}\bar{B}^2}{K_B^2(\bar{I}, \bar{T})} \frac{\partial K_B(\bar{I}, \bar{T})}{\partial T} + r_3 \qquad (2.36c)$$

$$\delta_1 + (1 - \pi)\nu\bar{B} > - \bar{B} \frac{\partial r_B(\bar{U})}{\partial U} \qquad (2.36d)$$

$$\nu_i > r_{i0}, \quad i = 1, 3. \qquad (2.36e)$$

$$\nu_2 > r_{20}\bar{I} \qquad (2.36f)$$

Then the equilibrium $\bar{E}$ is locally asymptotically stable.

The following lemma establishes the region of attraction for the system (2.33) whose proof is easy hence we omit it.

LEMMA 2.3.1 The set

$$\bar{R} = \left\{ (B,I,T,U,F_1,F_2,F_3) : 0 \leq B \leq K_a, \ 0 \leq I \leq I_a, \ 0 \leq T + U \leq Q_0/\delta \right.$$

$$\left. 0 \leq F_1 \leq r_1/\nu_1, \ 0 \leq F_2 \leq r_2 I_a/\nu_2, \ 0 \leq F_3 \leq r_3 Q_0/\nu_3\delta \right\} \text{ attracts}$$

all solution initiating in the interior of the positive octant, where

$$K_a = \frac{K_{BO}}{2} \left[1 + \left\{ 1 + 4r_1 r_{10}/\nu_1 r_{BO} K_{BO} \right\}^{1/2} \right], \quad \delta = \min(\delta_0,\delta_1),$$

$$I_a = (\beta K_a - \gamma_0)\gamma_1, \quad \beta K_a > \gamma_0.$$

In the following theorem we have shown that $\bar{E}$ is globally asymptotically stable under certain conditions.

THEOREM 2.3.2 In addition to the assumptions (2.2) and (2.3), let $r_B(U)$ and $K_B(I,T)$ satisfy in $\bar{R}$

$$\bar{K}_m \leq K_B(I,T) \leq K_{BO}, \quad 0 \leq -\frac{\partial r_B(U)}{\partial U} \leq \bar{\rho},$$

$$0 \leq -\frac{\partial K_B(I,T)}{\partial I} \leq \bar{k}_1, \quad 0 \leq -\frac{\partial K_B(I,T)}{\partial T} \leq \bar{k}_2, \tag{2.37}$$

for some positive constants $\bar{K}_m$, $\bar{\rho}$, $\bar{k}_1$, $\bar{k}_2$.

Then if the following inequalities hold

$$\left[\frac{r_{BO} K_a \bar{k}_1}{\bar{K}_m^2} + \alpha + \beta \right]^2 < \frac{4}{3} \frac{r_{BO} \gamma_1}{K_B(\bar{I},\bar{T})} \tag{2.38a}$$

$$\left[\frac{r_{BO} K_a \bar{k}_2}{\bar{K}_m^2} + \frac{\alpha_1 Q_0}{\delta} + \pi\nu\bar{U} \right]^2 < \frac{2}{3} \frac{r_{BO}}{K_B(\bar{I},\bar{T})} (\delta_0 + \alpha_1 \bar{B}) \tag{2.38b}$$

$$\left[\bar{\rho} + \frac{\nu Q_0}{\delta} + \alpha_1 \bar{T} \right]^2 < \frac{2}{3} \frac{r_{BO}}{K_B(\bar{I},\bar{T})} (\delta_1 + \nu\bar{B}) \tag{2.38c}$$

$$\left[\pi \, \nu \, + \, \alpha_1\right]^2 K_a^2 \, < \, (\delta_0 \, + \, \alpha_1 \bar{B})(\delta_1 \, + \, \nu \bar{B}) \qquad (2.38d)$$

$\bar{E}$ is globally asymptotically stable with respect to all solutions initiating in the interior if the positive octant.

Proof: Consider the following positive definite function about $\bar{E}$,

$$W(B,I,T,U,F_1,F_2,F_3) = (B - \bar{B} - \bar{B} \ln(B/\bar{B})) + (I - \bar{I} - \bar{I} \ln(I/\bar{I}))$$

$$+ \frac{1}{2}(T - \bar{T})^2 + \frac{1}{2}(U - \bar{U})^2 + \frac{1}{2}c_1(F_1 - \bar{F}_1)^2$$

$$+ \frac{1}{2}c_2(F_2 - \bar{F}_2)^2 + \frac{1}{2}c_3(F_3 - \bar{F}_3)^2 \qquad (2.39)$$

where c_1, c_2, c_3 are positive constants to be chosen suitably. Differentiating W with respect to t along the solution of the system (3.33) and choosing the constants $c_1 = r_{10}K_{B0}/r_1\bar{B}$, $c_2 = r_{20}/r_2$, $c_3 = r_{30}/r_3$, one can see that $\frac{dW}{dt}$ is negative definite under the conditions (2.38). Hence W is a Liapunov function with respect to $\bar{E}$, whose domain contains the region of attraction $\bar{R}$, proving the theorem.

The above theorem says that the forestry resource can be maintained at an appropriate level by suitable efforts for reforestation of the species and by controlling the industrialization and pollution.

CASE II: $Q = Q(I)$

In this case also it can be checked that the only equilibrium of the model (2.33) is $\overset{\approx}{E}(\overset{\approx}{B},\overset{\approx}{I},\overset{\approx}{T},\overset{\approx}{U},\overset{\approx}{F}_1,\overset{\approx}{F}_2,\overset{\approx}{F}_3)$ and it exists, provided

$$\frac{\partial r_B}{\partial U}\frac{d\hat{h}}{dB} - \frac{r_{B0}B}{K_B(I,\hat{g}(B))} + \frac{r_{B0}B}{K_B^2(I,\hat{g}(B))}\frac{\partial K_B}{\partial T}\frac{d\hat{g}}{dB} - \frac{r_1 r_{10}}{\nu_1 B^2} < 0 \quad \text{and}$$

$$r_2 r_{20} I_c > \nu_2 \gamma_0 \qquad (2.40)$$

where $\hat{h}(B) = \alpha\hat{Q}(B)B/[\delta_0^*\delta_1 + (\delta_0^*\nu + \delta_1\alpha_1)B + (1 - \pi)\nu\alpha_1 B^2]$,

$\hat{g}(B) = [\hat{Q}(B) + \pi\nu B \hat{h}(B)]/(\delta_0^* + \alpha_1 B)$, $\hat{Q}(B) = Q(\hat{f}(B)) + r_3 r_{30} T_c/\nu_3$,

$\hat{f}(B) = [\beta\nu_2 B + r_2 r_{20} I_c - \nu_2\gamma_0]/[\gamma_1\nu_2 + r_2 r_{20}]$, $\delta_0^* = \delta_0 + r_3 r_{30}/\nu_3$

The following theorem gives the criteria for local stability of $\overset{\approx}{E}$ whose proof is similar to theorem 2.3.1 and hence is omitted.

THEOREM 2.3.3 Let the following inequalities hold.

$$\frac{r_{BO}\overset{\approx}{B}}{K_B(\overset{\approx}{I},\overset{\approx}{T})} + \frac{r_{10}\overset{\approx}{F_1}}{\overset{\approx}{B}} > \beta\overset{\approx}{I} + 2\alpha_1\overset{\approx}{T} - (1 + \pi)\nu\overset{\approx}{U} + \frac{r_1}{K_{BO}} \qquad (2.41a)$$

$$\gamma_1\overset{\approx}{I} > Q'(\overset{\approx}{I}) + \alpha\overset{\approx}{B} - \frac{r_{BO}\overset{\approx}{B}^2}{K_B^2(\overset{\approx}{I},\overset{\approx}{T})} \frac{\partial K_B(\overset{\approx}{I},\overset{\approx}{T})}{\partial I} + r_2 \qquad (2.41b)$$

$$\delta_0 > - \frac{r_{BO}\overset{\approx}{B}^2}{K_B^2(\overset{\approx}{I},\overset{\approx}{T})} \frac{\partial K_B(\overset{\approx}{I},\overset{\approx}{T})}{\partial T} + r_3 \qquad (2.41c)$$

$$\delta_1 + (1 - \pi)\nu\overset{\approx}{B} > - \overset{\approx}{B} \frac{\partial r_B(\overset{\approx}{U})}{\partial U} \qquad (2.41d)$$

$$\nu_i > r_{i0}, \quad i = 1, 3. \qquad (2.41e)$$

$$\nu_2 > r_{20}\overset{\approx}{I} \qquad (2.41f)$$

Then the equilibrium $\overset{\approx}{E}$ is locally asymptotically stable.

The following lemma establishes region of attraction for the system (2.33).

LEMMA 2.3.2 The set $\overset{\approx}{\mathbb{R}} = \{(B,I,T,U,F_1,F_2,F_3) : 0 \leq B \leq K_a, 0 \leq I \leq I_a, 0 \leq T + U \leq Q_a/\delta, 0 \leq F_1 \leq r_1/\nu_1, 0 \leq F_2 \leq r_2 I_a/\nu_2, 0 \leq F_3 \leq r_3 Q_a/\nu_3\delta\}$ attracts all solution initiating in the interior of the positive octant, where

$$K_a = \frac{K_{BO}}{2} \left[1 + \left\{ 1 + 4r_1 r_{10}/\nu_1 r_{BO} K_{BO} \right\}^{1/2} \right], \quad \delta = \min(\delta_0, \delta_1),$$

$$Q_a = Q(I_a), \quad I_a = (\beta K_a - \gamma_0)/\gamma_1, \quad \beta K_a > \gamma_0.$$

In the following theorem we have shown that $\overset{\approx}{E}$ is globally asymptotically stable under certain conditions.

THEOREM 2.3.4   In addition to the assumptions (2.2a), (2.3) and (2.4), let $r_B(U)$, $K_B(I,T)$ and $Q(I)$ satisfy in $\overset{\approx}{\mathbb{R}}$

$$K_m \le K_B(I,T) \le K_{BO}, \quad 0 \le -\frac{\partial r_B(U)}{\partial U} \le \overset{\approx}{\rho}, \quad 0 \le Q'(I) \le \overset{\approx}{k}$$

$$0 \le -\frac{\partial K_B(I,T)}{\partial I} \le \overset{\approx}{k}_1, \quad 0 \le -\frac{\partial K_B(I,T)}{\partial T} \le \overset{\approx}{k}_2, \tag{2.42}$$

for some positive constants $\overset{\approx}{K}_m$, $\overset{\approx}{\rho}$, $\overset{\approx}{k}$, $\overset{\approx}{k}_1$, $\overset{\approx}{k}_2$. Then if the following inequalities hold

$$\left[\frac{r_{BO} K_a \overset{\approx}{k}_1}{\overset{\approx}{K}_m^2} + \alpha + \beta \right]^2 < \frac{2}{3} \frac{r_{BO} \gamma_1}{\overset{\approx}{K}_B(I,T)} \tag{2.43a}$$

$$\left[\frac{r_{BO} K_a \overset{\approx}{k}_2}{\overset{\approx}{K}_m^2} + \frac{\alpha_1 Q_a}{\delta} + \pi \nu \overset{\approx}{U} \right]^2 < \frac{4}{9} \frac{r_{BO}}{\overset{\approx}{K}_B(I,T)} (\delta_0 + \alpha_1 \overset{\approx}{B}) \tag{2.43b}$$

$$\left[\overset{\approx}{\rho} + \frac{\nu Q_a}{\delta} + \alpha_1 \overset{\approx}{T} \right]^2 < \frac{2}{3} \frac{r_{BO}}{\overset{\approx}{K}_B(I,T)} (\delta_1 + \nu \overset{\approx}{B}) \tag{2.43c}$$

$$\left[\pi \nu + \alpha_1 \right]^2 K_a^2 < \frac{2}{3} (\delta_0 + \alpha_1 \overset{\approx}{B})(\delta_1 + \nu \overset{\approx}{B}) \tag{2.43d}$$

$$\overset{\approx}{k}^2 < \frac{2}{3} \gamma_1 (\delta_0 + \alpha_1 \overset{\approx}{B}) \tag{2.43e}$$

$\overset{\approx}{E}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive octant.

The above theorem implies that if conservation efforts to control industrialization and pollution are applied, the resource

can be maintained at a desired level even in this case also, though more effort is needed to control the industrialization.

2.4 EXAMPLES

To explain the applicability of the results discussed above, we give the the following two examples by choosing the functions $r_B(U)$ and $K_B(I,T)$ as

$$r_B(U) = r_{B0} - r_{B1}U \tag{2.44a}$$

$$K_B(I,T) = K_{B0} - K_{B1}I - K_{B2}T \tag{2.44b}$$

where the coefficients are positive.

Example 1. In this example, we choose the following set of parameters to show that the interior equilibrium $E^*(B^*,I^*,T^*,U^*)$ of the model (2.1) exists and is globally asymptotically stable.
$\gamma_0 = 0.1$, $\gamma_1 = 3.0$, $\alpha = 0.1$, $\beta = 0.2$, $\alpha_1 = 0.02$, $\nu = 0.03$,
$\pi = 0.5$, $\delta_0 = 5.0$, $\delta_1 = 6.0$, $Q_0 = 30.0$, $r_{B0} = 3.8$, $r_{B1} = 0.04$,
$K_{B0} = 18.3279$, $K_{B1} = 0.01$, $K_{B2} = 0.02$.

By choosing the above set of parameters in the model (2.1), it can be checked that the interior equilibrium E^* exists and its coordinates are given by

$B^* \simeq 17.6$, $I^* \simeq 1.14$, $T^* \simeq 5.6203$, $U^* \simeq 0.3031$.

It can also be checked that the conditions (2.18) in theorem 2.2.2 are satisfied and hence E^* is locally asymptotically stable.

By choosing $K_m = 2$ in theorem 2.2.3, it can further be checked that the conditions (2.20) in this theorem are satisfied. This shows that E^* is globally asymptotically stable.

Example 2.

In this example, we choose the following set of parameters to show that the interior equilibrium $\bar{E}(\bar{B}, \bar{I}, \bar{T}, \bar{U}, \bar{F}_1, \bar{F}_2, \bar{F}_3)$ of the model (2.33) exists and is globally asymptotically stable.

$\gamma_0 = 0.1$, $\gamma_1 = 3.0$, $\alpha = 0.1$, $\beta = 0.2$, $\alpha_1 = 0.02$, $\nu = 0.03$,

$\pi = 0.5$, $\delta_0 = 5.0$, $\delta_1 = 6.0$, $Q_0 = 30.0$, $\nu_1 = 0.14$, $\nu_2 = 0.15$,

$\nu_3 = 0.15$, $r_{10} = 0.11$, $r_{20} = 0.1$, $r_{30} = 0.12$, $r_1 = 0.3$, $r_2 = 0.08$,

$r_3 = 1.0$, $r_{B0} = 20.9347$, $r_{B1} = 0.04$, $K_{B0} = 18.3279$, $K_{B1} = 0.01$,

$K_{B2} = 0.02$, $I_c = 1.0$, $T_c = 0.5$.

With the the above set of parameters in the model (2.33) and the functions given by (2.44), it can be checked that the interior equilibrium $\bar{E}$ exists and its coordinates are given by

$\bar{B} \simeq 18.1$, $\quad \bar{I} \simeq 1.1965$, $\quad \bar{T} \simeq 5.1082$, $\quad \bar{U} \simeq 0.2826$,

$\bar{F}_1 \simeq 0.0266$, $\quad \bar{F}_2 \simeq 0.1048$, $\quad \bar{F}_3 \simeq 30.7213$.

It can also be checked that the conditions (2.36) in theorem 2.3.1 are satisfied and hence $\bar{E}$ is locally asymptotically stable.

Further by selecting $\bar{K}_m = 3$ in theorem 2.3.2, it can be checked that the conditions (2.38) in this theorem are satisfied showing that $\bar{E}$ is globally asymptotically stable.

CONCLUSIONS :

In this chapter, a mathematical model to study the effect of pollutant(toxicant) caused by industrialization on biological species such as plant population in a forest stand (forestry resource biomass) is presented. It is assumed that the growth rate of plant population depends upon the uptake concentration of the toxicant present in the plant population but the corresponding carrying capacity may depend upon both the the density of industrialization and the concentration of toxicant in the

environment. It is assumed that the dynamics of the species is governed by the generalized logistic equation and the dynamics of industrialization follows predation type equation. By analysing the model, it is shown that when the pollutant is exhausted in the environment by an instantaneous introduction, the forestry resource may recover back to a level the magnitude of which depends upon the equilibrium level of industrialization. However, when the pollutant is exhausted in the environment with a constant rate, the forestry resource biomass density will settle down to a lower equilibrium level than the case of instantaneous introduction of pollutant as well as the case when industrialization is absent. In the case of industrialization dependent introduction of pollutant, it is shown that the system will again settle down to a steady state whose magnitude would depend upon influx and washout rate of pollutant present in the environment, influx rate being dependent upon the steady state industrialization. In the case of periodic introduction of pollutant, it is shown that for small amplitude the stability behavior of the system is same as that of the constant introduction of pollutant in the environment. In the conservation model it is shown that by conserving the resource biomass, and by controlling the pressures of industrialization and pollution on the resource biomass an appropriate level of the forestry resource can be maintained.

Fig. 2.1

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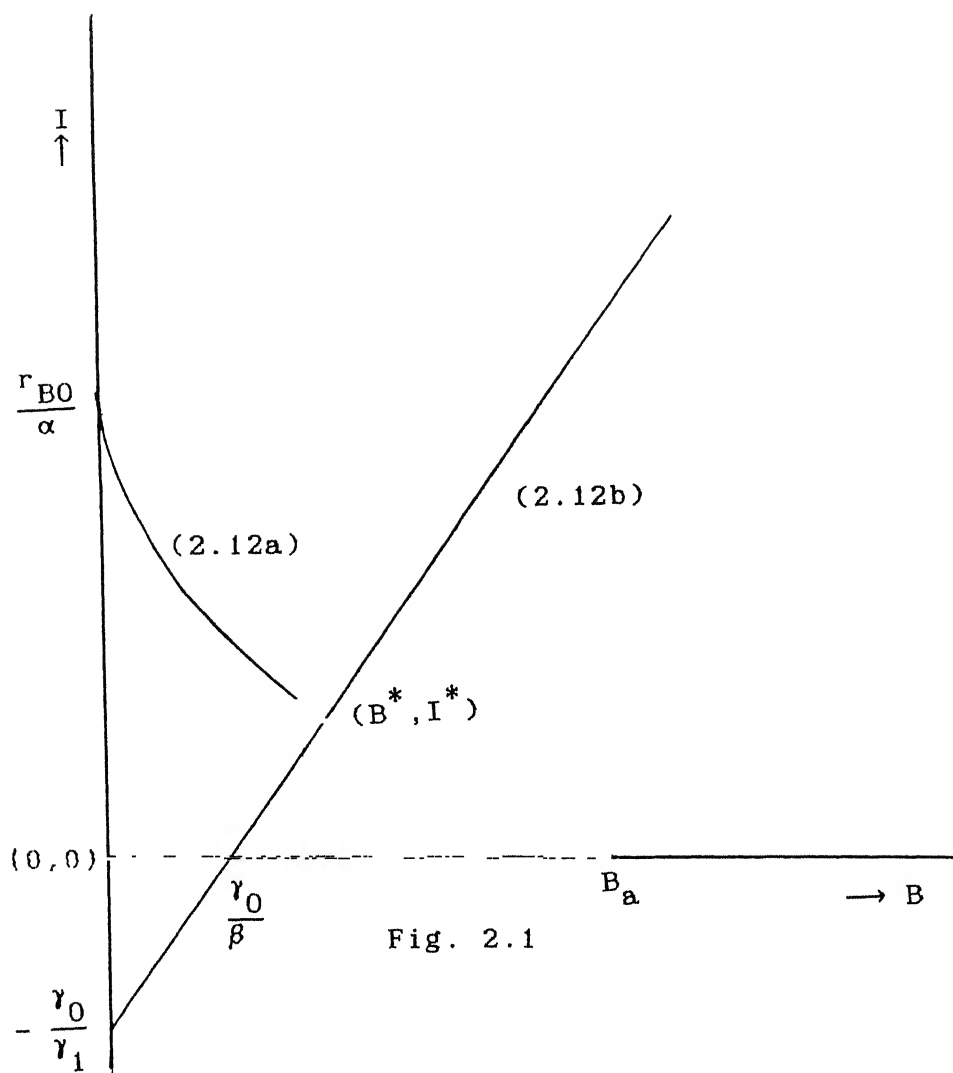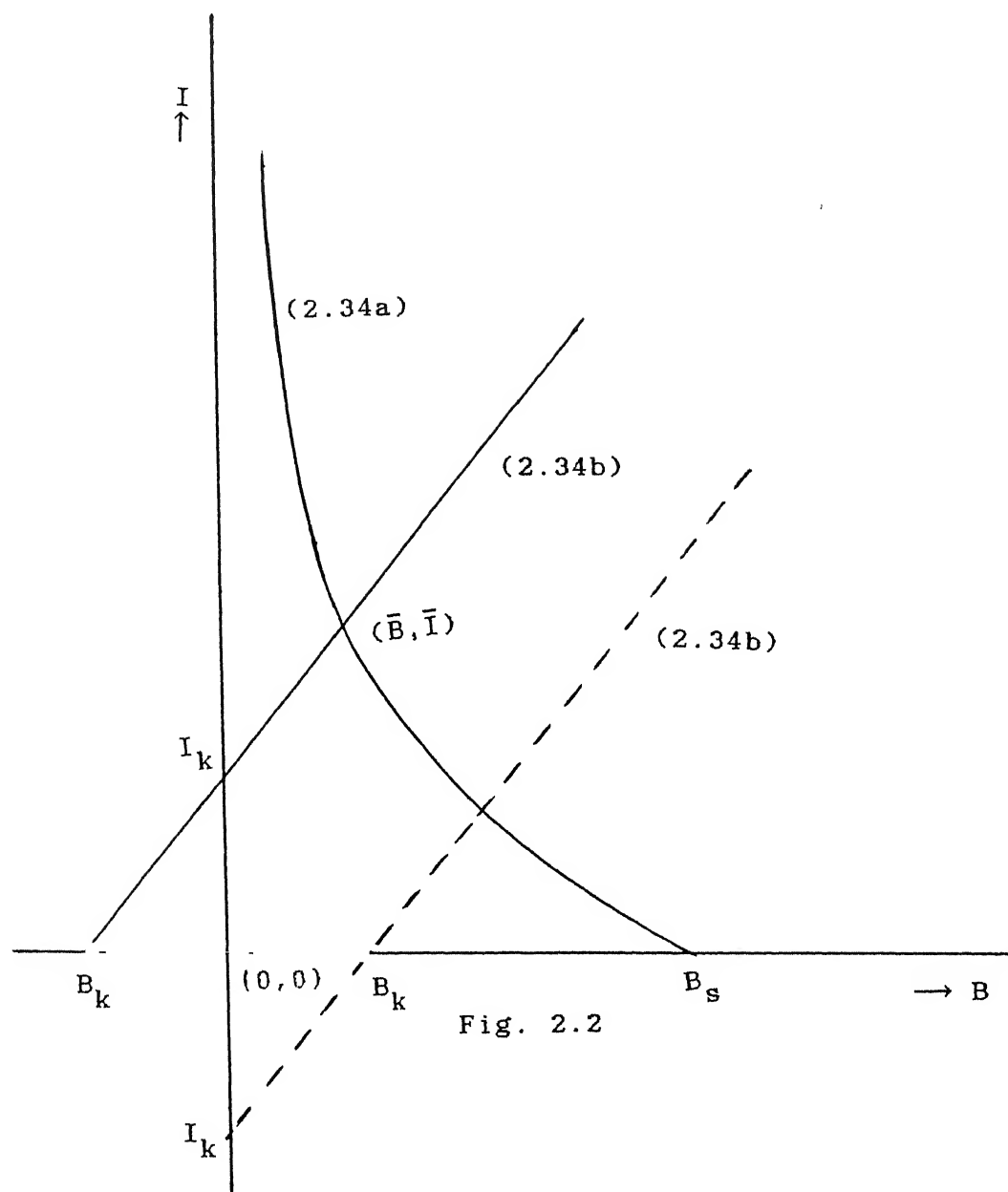
Fig. 2.2

CHAPTER III

MODELLING THE DEPLETION AND CONSERVATION OF FORESTRY RESOURCES: EFFECTS OF INDUSTRIALIZATION AND POPULATION

3.0 INTRODUCTION

The depletion of forestry resources, though a world wide phenomenon, is a major cause of concern in developing countries due to conflicting interests between sustainable development and rapid industrialization. In these countries, the forestry resources are being depleted, due to their use by wood processing industries and by populations using forest land for resettlement and colonization, for expansion of agricultural land and cutting of trees for fuel and fodder, leading to various ecological and environmental consequences, Kormondy (1986), Lamberson (1988). There are many ecological regions around the world where forest has been cut without any ecological and environmental considerations and one such region is Doon Valley in the northern part of Uttar Pradesh in India. The report, prepared under the auspices of International Institute for Applied Systems Analysis, suggests that the main reasons for the decrease of forestry resources are lime stone quarrying, wood based industries, growth of human and live stock populations etc., Munn and Fedorov (1986). To avoid depletion of forests and to maintain the ecological stability of the region it is essential to study the effects of industrialization and population growth on the forestry resources and to suggest the measures for their conservation.

It may be noted here that a very little effort has been made to study the depletion of forest caused by industrialization and

population using mathematical model. Shukla et al. (1989) have studied the depletion of forest resource biomass by considering the cumulative effects of industrialization and population on resource but they have not considered the effects of these factors separately. It is instructive to point out here that the growth of industrialization may wholly depend upon forestry resource and should be modelled as predation process. Here the term forestry resource is used for all forestry resources such as plant/tree population and wildlife population.

In this chapter we, therefore, propose a dynamic model for the depletion of forestry resource biomass by taking into account of the effects of increasing industrialization and population pressures. It is assumed that growth rate of the forest biomass density depends upon the population density while its carrying capacity depends upon the density of population and industrialization and decrease as the densities of population and industrialization increase. It is also assumed that growth rates of forest biomass and population density are governed by logistic type equations, Quaddus (1985). It is further assumed that industrialization depends wholly on the resource and the interaction process is of predator type.

We assume that all functions utilized are sufficiently smooth and the solutions to the initial value problems exist uniquely and are continuable for all positive time. Stability Analysis (La Salle and Lefschetz (1961)) is used to analyse the model.

3.1 MATHEMATICAL MODEL

We consider a forest ecosystem in which the forestry resource biomass is depleted continuously due to industrialization and

population pressures. Keeping in view the discussion above, the model can be written by the following system of autonomous differential equations :

$$\frac{dB}{dt} = r_B(P)B - \frac{r_{B0} B^2}{K_B(I,P)} - \alpha IB$$

$$\frac{dI}{dt} = - \gamma_0 I - \gamma_1 I^2 + \beta IB \qquad (3.1)$$

$$\frac{dP}{dt} = r_P(I,B)P - \frac{r_{P0} P^2}{L}$$

$B(0) = B_0 \geq 0, \ I(0) = I_0 \geq 0, \ P(0) = P_0 \geq 0.$

Here $B(t)$ is the density of the resource biomass, $I(t)$ the density of industrialization and $P(t)$ the density of population at time t. L is the maximum population density which the environment can support and it is a positive constant, γ_0 is the natural depletion rate coefficient and γ_1 is the intraspecific coefficient of industrialization, β is the growth rate coefficient of industrialization due to the depletion rate coefficient α of the resource biomass. The coefficients α, β, γ_0, γ_1 are assumed to be positive.

In our model (3.1), the function $r_B(P)$ is the growth rate coefficient of forest biomass which decreases as P increase and hence we assume

$$r_B(0) = r_{B0} > 0, \quad \frac{\partial r_B(P)}{\partial P} < 0 \quad \text{for } P \geq 0$$

and $r_B(\bar{P}) = 0$ for some $\bar{P} > 0$ \qquad (3.2)

Similarly the function $K_B(I,P)$ denotes the maximum resource biomass density which the environment can support and it decreases as I and P increase and hence we assume

$$K_B(0,0) = K_{B0} > 0, \quad \frac{\partial K_B(I,P)}{\partial I} < 0, \quad \frac{\partial K_B(I,P)}{\partial P} < 0 \quad \text{for } I \geq 0, P \geq 0$$
(3.3)

The function $r_P(I,B)$ is the growth rate coefficient of the population density which increases as I and B increase and hence we assume

$$r_P(0,0) = r_{P0} > 0, \quad \frac{\partial r_P(I,B)}{\partial I} > 0, \quad \frac{\partial r_P(I,B)}{\partial B} > 0 \quad \text{for } I \geq 0, B \geq 0$$
(3.4)

3.2 EXISTENCE OF EQUILIBRIA

The equilibria of the model (3.1) is given by

$$r_B(P)B - \frac{r_{B0} B^2}{K_B(I,P)} - \alpha IB = 0$$
(3.5a)

$$\gamma_0 I - \gamma_1 I^2 + \beta IB = 0$$
(3.5b)

$$r_P(I,B)P - \frac{r_{P0} P^2}{L} = 0$$
(3.5c)

Solving the above equations, we get six equilibria, namely $E_0(0,0,0)$, $E_1(K_{B0},0,0)$, $E_2(0,0,L)$, $E_3(\tilde{B},\tilde{I},0)$, $E_4(\approx{B},0,\approx{P})$ and $E^*(B^*,I^*,P^*)$.

It is clear that E_0, E_1 and E_2 always exist. We will show the existence of other three equilibria as follows.

Existence of $E_3(\tilde{B},\tilde{I},0)$:

Here $\tilde{B}$ and $\tilde{I}$ are the positive solution of the system of equations

$$\alpha I = r_{B0}(1 - \frac{B}{K_B(I,0)})$$
(3.6a)

$$\gamma_1 I = \beta B - \gamma_0$$
(3.6b)

From (3.6a) we note that I is a decreasing function of B starting from r_{B0}/α and it intersects the B axis at K_{B0}. From

(3.6b) we note that it is an equation of straight line passing through $(0, -\gamma_0/\gamma_1)$ and $(\gamma_0/\beta, 0)$. Hence the above two isoclines intersect at a unique point $(\tilde{B}, \tilde{I})$ iff

$$K_{BO} > \gamma_0/\beta \tag{3.7}$$

Existence of $E_4(\overset{\approx}{B}, 0, \overset{\approx}{P})$:

Here $\overset{\approx}{B}$ and $\overset{\approx}{P}$ are the positive solution of the system of algebraic equations

$$r_B(P)K_B(0,P) = r_{BO}B \tag{3.8a}$$

$$r_{PO}P = L\, r_P(0,B) \tag{3.8b}$$

It is easy to check that in the isoclines (3.8a) P is a decreasing function of B starting from $\bar{P}$ and in the isocline (3.8b) P is a increasing function of B starting from L. Thus the two isoclines (3.8a) and (3.8b) will intersect at a unique point $(\overset{\approx}{B}, \overset{\approx}{P})$ iff

$$L < \bar{P} \tag{3.9}$$

Existence of $E(B^*, I^*, P^*)$:

Here B^*, I^* and P^* are the positive solution of the system of equations:

$$\alpha I = r_B(P) - \frac{r_{BO}\, B}{K_B(I,P)} \tag{3.10a}$$

$$B = (\gamma_1 I + \gamma_0)/\beta = f(I) , \quad (\text{say}) \tag{3.10b}$$

$$P = \frac{r_P(I,B)L}{r_{PO}} \tag{3.10c}$$

We see that f(I) increases as I increases and hence B also increases with I.

Substituting B from (3.10b) in (3.10a) and (3.10c), we get

$$\alpha I = r_B(P) - \frac{r_{BO} f(I)}{K_B(I,P)} \tag{3.11}$$

$$P = \frac{r_P(I, f(I)) L}{r_{PO}} \tag{3.12}$$

From (3.11) we note the following:

$$\text{When } P \rightarrow 0, \quad I \rightarrow I_a \tag{3.13a}$$

where I_a is given by

$$\alpha I_a = r_{BO}\left(1 - \frac{f(I_a)}{K_B(I_a, 0)}\right) \tag{3.13b}$$

'or I_a to be positive we must have

$$K_B(I_a, 0) > \gamma_0/\beta \tag{3.13c}$$

t is noted here that (3.13c) always implies (3.7) but not onversely.

$$\text{When } I \rightarrow 0, \quad P \rightarrow P_a \tag{3.13d}$$

1ere P_a is the zero of

$$F(P) = \gamma_0 r_{BO} - \beta r_B(P) K_B(0, P) \tag{3.13e}$$

note that

$$F(0) = r_{BO}(\gamma_0 - \beta K_{BO}) < 0 \text{ for } K_{BO} > \gamma_0/\beta$$

$$F(\bar{P}) = \gamma_0 r_{BO} > 0$$

ce there is a P_a in the interval $0 < P_a < \bar{P}$ such that $F(P_a) = 0$

lso have from (3.11)

$$x + \frac{\gamma_1 r_{BO}}{\beta K_B(I,P)} - \frac{r_{BO} f(I)}{K_B^2(I,P)} \frac{\partial K_B}{\partial I}\right] = \frac{\partial r_B(P)}{\partial P} + \frac{r_{BO} f(I)}{K_B^2(I,P)} \frac{\partial K_B}{\partial P} \tag{3.13f}$$

lows that $\frac{dI}{dP} < 0$.

the above analysis, we note that the isocline (3.11) is

ng function of P starting from I_a and it intersects the

P-axis at P_a.

From (3.12) we note that P is an increasing function of I starting from L_a, where L_a is given by

$$L_a = L \ r_P(0,\gamma_0/\beta)/r_{PO} \qquad (3.14)$$

Thus the isoclines (3.11) and (3.12) must intersect at a unique point [see fig. 3.1], provided

$$L_a < P_a \qquad (3.15)$$

The intersection value of the above two isoclines gives the I-P coordinates of E^* and then its B coordinate can be computed from (3.10b). This completes the existence of E^*.

REMARK: From (3.6a), (3.8a) and (3.10a) comparing the value of B we note that the magnitude of the forest biomass at equilibrium level in the case when industrialization and population are acting simultaneously on the resource is lower than the case when either industrialization or population pressure acts on it.

3.3 STABILITY ANALYSIS

To study the local stability behavior of equilibria, we compute the variational matrices corresponding to each equilibrium. Using the analogous notations for the variational matrices i.e. M_1 is the variational matrix corresponding to E_1, we have

$$M_0 = \begin{bmatrix} r_{BO} & 0 & 0 \\ 0 & -\gamma_0 & 0 \\ 0 & 0 & r_{PO} \end{bmatrix}$$

$$
M_1 = \begin{bmatrix} -r_{B0} & r_{B0}\dfrac{\partial K_B(0,0)}{\partial I} - \alpha K_{B0} & r_{B0}\dfrac{\partial K_B(0,0)}{\partial P} + K_{B0}\dfrac{\partial r_B(0)}{\partial P} \\[2ex] 0 & \beta K_{B0} - \gamma_0 & 0 \\[2ex] 0 & 0 & r_P(0,K_{B0}) \end{bmatrix}
$$

$$
M_2 = \begin{bmatrix} r_B(L) & 0 & 0 \\[2ex] 0 & -\gamma_0 & 0 \\[2ex] L\dfrac{\partial r_P(0,0)}{\partial B} & L\dfrac{\partial r_P(0,0)}{\partial I} & -r_{P0} \end{bmatrix}
$$

$$
M_3 = \begin{bmatrix} -\dfrac{r_{B0}\tilde{B}}{K_B(\tilde{I},0)} & F_{12} & F_{13} \\[2ex] \beta\tilde{I} & -\gamma_1\tilde{I} & 0 \\[2ex] 0 & 0 & r_P(\tilde{I},\tilde{B}) \end{bmatrix}
$$

where
$$ F_{12} = \frac{r_{B0}\tilde{B}^2}{K_B^2(\tilde{I},0)}\frac{\partial K_B(\tilde{I},0)}{\partial I} - \alpha\tilde{B} \tag{3.16a} $$

$$ F_{13} = \frac{r_{B0}\tilde{B}^2}{K_B^2(\tilde{I},0)}\frac{\partial K_B(\tilde{I},0)}{\partial P} + \tilde{B}\frac{\partial r_B(0)}{\partial P} \tag{3.16b} $$

$$
M_4 = \begin{bmatrix}
- r_B(\overset{\approx}{P}) & G_{12} & G_{13} \\
0 & \beta \overset{\approx}{B} - \gamma_0 & 0 \\
\overset{\approx}{P} \dfrac{\partial r_P(0,\overset{\approx}{B})}{\partial B} & \overset{\approx}{P} \dfrac{\partial r_P(0,\overset{\approx}{B})}{\partial I} & - r_P(0,\overset{\approx}{B})
\end{bmatrix}
$$

where
$$
G_{12} = \frac{r_{B0}\overset{\approx}{B}{}^2}{K_B^2(0,\overset{\approx}{P})} \frac{\partial K_B(0,\overset{\approx}{P})}{\partial I} - \alpha \overset{\approx}{B} \tag{3.17a}
$$

$$
G_{13} = \frac{r_{B0}\overset{\approx}{B}{}^2}{K_B^2(0,\overset{\approx}{P})} \frac{\partial K_B(0,\overset{\approx}{P})}{\partial P} + \overset{\approx}{B} \frac{\partial r_B(\overset{\approx}{P})}{\partial P} \tag{3.17b}
$$

$$
M^* = \begin{bmatrix}
- \dfrac{r_{B0}B^*}{K_B(I^*,P^*)} & H_{12} & H_{13} \\
\beta I^* & - \gamma_1 I^* & 0 \\
P^* \dfrac{\partial r_P(I^*,B^*)}{\partial B} & P^* \dfrac{\partial r_P(I^*,B^*)}{\partial I} & - \dfrac{r_{P0}P^*}{L}
\end{bmatrix}
$$

where
$$
H_{12} = \frac{r_{B0}B^{*2}}{K_B^2(I^*,P^*)} \frac{\partial K_B(I^*,P^*)}{\partial I} - \alpha B^* \tag{3.18a}
$$

$$
H_{13} = \frac{r_{B0}B^{*2}}{K_B^2(I^*,P^*)} \frac{\partial K_B(I^*,P^*)}{\partial P} + B^* \frac{\partial r_B(P^*)}{\partial P} \tag{3.18b}
$$

We note from M_0 that E_0 is a saddle point with stable manifold locally in I direction and unstable manifold locally in B-P plane. From M_1 we note that E_1 is also a saddle point with stable manifold locally in B direction and unstable manifold

locally in I-P plane. From M_2 we note that E_2 is also a saddle point with stable manifold locally in I-P plane and unstable manifold locally in B direction. From M_3 we note that E_3 is also a saddle point with stable manifold locally in B-I plane and unstable manifold locally in P direction. From M_4 we note that E_4 is also a saddle point whose stable manifold is locally in B-P plane and whose unstable manifold is locally in I direction.

The stability behavior of E^* is not obvious from M^*. However, in the following theorem we have found sufficient conditions for E^* to be locally asymptotically stable.

THEOREM 3.3.1   Let the following inequalities hold

$$\frac{r_{BO}B^*}{K_B(I^*,P^*)} > \beta I^* + P^* \frac{\partial r_P(I^*,B^*)}{\partial B} \qquad (3.19a)$$

$$\gamma_1 I^* > \alpha B^* + P^* \frac{\partial r_P(I^*,B^*)}{\partial I} - \frac{r_{BO}B^{*2}}{K_B^2(I^*,P^*)} \frac{\partial K_B(I^*,P^*)}{\partial I} \qquad (3.19b)$$

$$\frac{r_{PO}P^*}{L} > - B^* \frac{\partial r_B(P^*)}{\partial P} - \frac{r_{BO}B^{*2}}{K_B^2(I^*,P^*)} \frac{\partial K_B(I^*,P^*)}{\partial P} \qquad (3.19c)$$

Then E^* is locally asymptotically stable.

Proof :   If the inequalities (3.19) hold, then by Gershgorin's theorem (Lancaster and Tismanetsky, p.371, 1985) all eigen values of M^* will have negative real parts, and the theorem follows.

In the following theorem, we have shown that E^* is globally asymptotically stable under certain conditions. We first require a lemma which establishes the region of attraction for our system. The ideas used here are developed in Hsu (1978) and Freedman (1987).

LEMMA 3.3.1 The set

$$\mathbb{R} = \left\{(B,I,P): 0 \le B \le K_{BO}, \ 0 \le I \le I_m, \ 0 \le P \le P_m\right\} \text{ is a}$$

region of attraction for all solutions initiating in the positive

octant, where $I_m = (\beta K_{BO} - \gamma_0)/\gamma_1$, $P_m = L \ r_P(I_m, K_{BO})/r_{PO}$ and

$\beta K_{BO} > \gamma_0$.

Proof : From (3.1) we have

$$\frac{dB}{dt} = r_B(P)B - \frac{r_{BO} \ B^2}{K_B(I,P)} - \alpha IB$$

$$\le r_{BO} \ B - \frac{r_{BO} \ B^2}{K_{BO}}$$

hence $\lim_{t \to \infty} B(t) \le K_{BO}$

Again we have

$$\frac{dI}{dt} = - \gamma_0 I - \gamma_1 I^2 + \beta IB$$

$$\le (\beta K_{BO} - \gamma_0)I(1 - I/I_m)$$

hence $\lim_{t \to \infty} I(t) \le I_m$

Finally we have

$$\frac{dP}{dt} = r_P(I,B)P - \frac{r_{PO} \ P^2}{L}$$

$$\le r_P(I_m, K_{BO})P \left[1 - \frac{P}{P_m}\right]$$

hence $\lim_{t \to \infty} P(t) \le P_m$, proving the lemma.

THEOREM 3.3.2 In addition to the assumptions (3.2) — (3.4), let

$r_B(P)$, $K_B(I,P)$, $r_P(I,B)$ satisfy in $\mathbb{R}$

$$K_m \leq K_B(I,P) \leq K_{BO} \quad , \qquad 0 \leq - \frac{\partial r_B(P)}{\partial P} \leq \rho \quad ,$$

$$0 \leq \frac{\partial r_P(I,B)}{\partial I} \leq \rho_1 \quad , \qquad 0 \leq \frac{\partial r_P(I,B)}{\partial B} \leq \rho_2 \quad , \qquad (3.20)$$

$$0 \leq - \frac{\partial K_B(I,P)}{\partial I} \leq k_1 \quad , \qquad 0 \leq - \frac{\partial K_B(I,P)}{\partial P} \leq k_2$$

for some positive constants K_m, ρ, ρ_1, ρ_2, k_1, k_2. Then if the following inequalities hold

$$\left[\frac{r_{BO} \, K_{BO} \, k_1}{K_m^2} + \alpha + \beta \right]^2 < \frac{r_{BO} \, \gamma_1}{K_B(I^*,P^*)} \qquad (3.21a)$$

$$\left[\frac{r_{BO} \, K_{BO} \, k_2}{K_m^2} + \rho + \rho_2 \right]^2 < \frac{r_{BO} \, r_{PO}}{L K_B(I^*,P^*)} \qquad (3.21b)$$

$$\rho_1^2 < \frac{\gamma_1 r_{PO}}{L} \qquad (3.21c)$$

E^* is globally asymptotically stable with respect to all solutions initiating in the positive octant.

Proof: We consider the following positive definite function about E^*,

$$V(B,I,P) = \left(B - B^* - B^* \ln \frac{B}{B^*} \right) + \left(I - I^* - I^* \ln \frac{I}{I^*} \right)$$

$$+ \left(P - P^* - P^* \ln \frac{P}{P^*} \right) \qquad (3.22)$$

Differentiating V with respect to t along the solution of (3.1), we get

$$\frac{dV}{dt} = (B - B^*) \left[r_B(P) - \frac{r_{BO} \, B}{K_B(I,P)} - \alpha I \right] + (I - I^*) \left[- \gamma_0 - \gamma_1 I + \beta B \right]$$

$$+ (P - P^*) \left[r_P(I,B) - \frac{r_{PO} \, P}{L} \right]$$

A little computation yields

$$\frac{dV}{dt} = - \frac{r_{BO}}{K_B(I^*,P^*)} (B - B^*)^2 - \gamma_1 (I - I^*)^2 - \frac{r_{PO}}{L} (P - P^*)^2$$

$$+ (B - B^*)(I - I^*) \left[- r_{BO}B\xi_1(I,P) + \beta - \alpha \right]$$

$$+ (B - B^*)(P - P^*) \left[- r_{BO}B\xi_2(I^*,P) + \eta(P) + \eta_2(I^*,B) \right]$$

$$+ (I - I^*)(P - P^*) \left[\eta_1(I,B) \right] \qquad (3.23)$$

where

$$\eta(P) = \begin{cases} \left[r_B(P) - r_B(P^*) \right]/(P - P^*) , & P \neq P^* \\ \\ \dfrac{\partial r_B(P^*)}{\partial P} , & P = P^* \end{cases} \qquad (3.24a)$$

$$\eta_1(I,B) = \begin{cases} \left[r_P(I,B) - r_P(I^*,B)/(I - I^*) \right] , & I \neq I^* \\ \\ \dfrac{\partial r_P(I^*,B)}{\partial I} , & I = I^* \end{cases} \qquad (3.24b)$$

$$\eta_2(I^*,B) = \begin{cases} \left[r_P(I^*,B) - r_P(I^*,B^*)/(B - B^*) \right] , & B \neq B^* \\ \\ \dfrac{\partial r_P(I^*,B^*)}{\partial B} , & B = B^* \end{cases} \qquad (3.24c)$$

$$\xi_1(I,P) = \begin{cases} \left[\dfrac{1}{K_B(I,P)} - \dfrac{1}{K_B(I^*,P)} \right] \Big/ (I - I^*), & I \neq I^* \\[4mm] -\dfrac{1}{K_B^2(I^*,P)} \dfrac{\partial K_B(I^*,P)}{\partial I}, & I = I^* \end{cases} \qquad (3.24d)$$

$$\xi_2(I^*,P) = \begin{cases} \left[\dfrac{1}{K_B(I^*,P)} - \dfrac{1}{K_B(I^*,P^*)} \right] \Big/ (P - P^*), & P \neq P^* \\[4mm] -\dfrac{1}{K_B^2(I^*,P^*)} \dfrac{\partial K_B(I^*,P^*)}{\partial P}, & P = P^* \end{cases} \qquad (3.24e)$$

Using (3.20) and mean value theorem, we note that

$$|\eta(P)| \leq \rho, \quad |\eta_1(I,B)| \leq \rho_1, \quad |\eta_2(I^*,B)| \leq \rho_2,$$

$$|\xi_1(I,P)| \leq k_1/K_m^2, \quad |\xi_2(I^*,P)| \leq k_2/K_m^2. \qquad (3.25)$$

Now equation (3.23) can further be written as sum of the quadratics

$$\frac{dV}{dt} = -\frac{1}{2} a_{11}(B - B^*)^2 + a_{12}(B - B^*)(I - I^*) - \frac{1}{2} a_{22}(I - I^*)^2$$

$$-\frac{1}{2} a_{11}(B - B^*)^2 + a_{13}(B - B^*)(P - P^*) - \frac{1}{2} a_{33}(P - P^*)^2$$

$$-\frac{1}{2} a_{22}(I - I^*)^2 + a_{23}(I - I^*)(P - P^*) - \frac{1}{2} a_{33}(P - P^*)^2$$

$$(3.26)$$

where

$$a_{11} = \frac{r_{B0}}{K_B(I^*,P^*)}, \quad a_{22} = \gamma_1, \quad a_{33} = \frac{r_{P0}}{L},$$

$$a_{12} = - r_{B0} B \xi_1(I,P) + \beta - \alpha,$$

$$a_{13} = - r_{B0} B \xi_2(I^*,P) + \eta(P) + \eta_2(I^*,B),$$

$$a_{23} = \eta_1(I,B).$$

The sufficient conditions for $\frac{dV}{dt}$ to be negative definite are that

$$a_{12}^2 < a_{11} \, a_{22} \qquad\qquad\qquad (3.27a)$$

$$a_{13}^2 < a_{11} \, a_{33} \qquad\qquad\qquad (3.27b)$$

$$a_{23}^2 < a_{22} \, a_{33} \qquad\qquad\qquad (3.27c)$$

hold. We note that $(3.21a,b,c) \Rightarrow (3.27a,b,c)$ respectively. Thus V is a Liapunov function with respect to E^* whose domain contains the region $\mathbb{R}$, proving the theorem.

The above theorem shows that if the inequalities (3.21) hold, then the resource biomass will settle down to a steady state whose magnitude will depend upon the steady state of industrialization and population. It is also noted here that increase in the densities of industrialization and population causes decrease in the equilibrium density of resource biomass and this steady state equilibrium is lower than the case when only either industrialization or population affects the resource. It is further noted that if these pressures continue without control the biomass may be threatened to extinction.

3.4 CONSERVATION MODEL

The depletion of resource biomass caused by industrialization and population at an increasing rate has alarmed the entire world and the need of conservation is emphasized in all meetings related to sustainable resource use. The conservation of resource biomass using appropriate efforts such as plantation, irrigation, fencing, environment friendly technology for industrialization and control of population should be employed to achieve the ecological stability and sustainable development.

Let $F_1(t)$, $F_2(t)$ and $F_3(t)$ be the densities of efforts applied to conserve the biomass $B(t)$, and to control the industrialization $I(t)$ and the population $P(t)$ respectively. It is considered that $F_1(t)$ is proportional to the depleted level of biomass from its carrying capacity K_{BO}, $F_2(t)$ and $F_3(t)$ are proportional to the undesired level of industrialization and population densities respectively. Keeping the above in view the dynamics of the system can be governed by the following system of differential equations:

$$\frac{dB}{dt} = r_B(P)B - \frac{r_{BO} B^2}{K_B(I,P)} - \alpha IB + r_{10}F_1$$

$$\frac{dI}{dt} = -\gamma_0 I - \gamma_1 I^2 + \beta IB - r_{20}F_2 I$$

$$\frac{dP}{dt} = r_P(I,B)P - \frac{r_{PO} P^2}{L} - r_{30}F_3 P$$

$$\frac{dF_1}{dt} = r_1(1 - \frac{B}{K_{BO}}) - \nu_1 F_1 \tag{3.28}$$

$$\frac{dF_2}{dt} = r_2(I - I_c) - \nu_2 F_2$$

$$\frac{dF_3}{dt} = r_3(P - P_c) - \nu_3 F_3$$

$B(0) \geq 0$, $I(0) \geq 0$, $P(0) \geq 0$, $F_i(0) \geq 0$, $i = 1,2,3$.

Here $r_i > 0$ represents the constant growth rate coefficient of the effort F_i and $\nu_i \geq 0$ is its respective depreciation rate coefficient. r_{10} is the growth rate coefficient of the resource biomass due to effort F_1, and r_{20}, r_{30} are the depletion rate coefficients of $I(t)$, $P(t)$ due to efforts F_2, F_3 respectively. I_c and P_c are the critical values of I and P which are harmless to the biomass. Other notations in the above model have the same meaning as in the model (3.1).

It should be noted here that in our model (3.28), $I > I_c$ and $P > P_c$. If $I \leq I_c$ and $P \leq P_c$, then dF_i/dt (i=2,3) is negative showing that there is no need to control industrialization and population.

3.4.1 EXISTENCE OF EQUILIBRIUM

It can easily be seen that there exists only one interior equilibrium $\hat{E}(\hat{B}, \hat{I}, \hat{P}, \hat{F}_1, \hat{F}_2, \hat{F}_3)$ and its coordinates are the positive solution of the system of algebraic equations:

$$I = \frac{1}{\alpha} \left[r_B(h(B)) - \frac{r_{B0}B}{K_B(I, h(B))} + \frac{r_1 r_{10}}{\nu_1 B} - \frac{r_1 r_{10}}{\nu_1 K_{B0}} \right] \tag{3.29a}$$

$$I = g(B), \tag{3.29b}$$

$$P = h(B), \tag{3.29c}$$

$$F_1 = \frac{r_1}{\nu_1} \left(1 - \frac{B}{K_{B0}}\right), \quad F_2 = \frac{r_2}{\nu_2} (I - I_c), \quad F_3 = \frac{r_3}{\nu_3} (P - P_c) \tag{3.29d}$$

where

$$g(B) = \frac{\beta \nu_2 B + r_2 r_{20} I_c - \nu_2 \gamma_0}{\gamma_1 \nu_2 + r_2 r_{20}} \tag{3.30a}$$

$$h(B) = \frac{L (\nu_3 r_P(g(B), B) + r_3 r_{30} P_c)}{\nu_3 r_{P0} + r_3 r_{30} L} . \tag{3.30b}$$

From (3.29a) we note that

When $B \to 0$, $I \to \infty$ \hfill (3.31a)

When $I \to 0$, $B \to B_s$ \hfill (3.31b)

where B_s is a positive solution of

$$r_{B0}B = K_B(0, h(B)) r_B(h(B)) + K_B(0, h(B)) \frac{r_1 r_{10}}{\nu_1 B} \left(1 - \frac{B}{K_{B0}}\right) \tag{3.31c}$$

Taking

$$G(B) = r_{B0}B - K_B(0, h(B)) r_B(h(B)) - K_B(0, h(B)) \frac{r_1 r_{10}}{\nu_1 B} \left(1 - \frac{B}{K_{B0}}\right)$$

we note that $G(0) < 0$, $G(K_{BO}) > 0$ and hence there exists B_s in the interval $0 < B_s < K_{BO}$ such that $G(B_s) = 0$.

It can also be checked that $\frac{dI}{dB}$ computed from (3.29a) is negative.

Further the isocline (3.29b) is a straight line passing through the points $(0, I_k)$ and $(B_k, 0)$, where

$$I_k = \frac{r_2 r_{20} I_c - \nu_2 \gamma_0}{\gamma_1 \nu_2 + r_2 r_{20}} , \qquad (3.32a)$$

$$B_k = \frac{\nu_2 \gamma_0 - r_2 r_{20} I_c}{\beta \nu_2} . \qquad (3.32b)$$

It should be noted here that I_k and B_k are of opposite sign.

If we take $I_k > 0$, then $B_k < 0$ and in this case the isoclines (3.29a) and (3.29b) will intersect at a unique point [see fig. 3.2].

If we take $I_k < 0$, then $B_k > 0$ and in this case the isoclines (3.29a) and (3.29b) will intersect at a unique point [see fig. 3.2] iff

$$B_k < B_s \qquad (3.33)$$

The intersection value of these two isoclines (3.29a) and (3.29b) will give the B-I coordinate of $\hat{E}$ and its other coordinates can be computed from (3.29c,d). It should be remarked here that for $\hat{F}_1$ to be positive, we must have

$$\hat{B} < K_{BO} \qquad (3.34)$$

3.4.2 STABILITY ANALYSIS

By computing the variational matrix corresponding to $\hat{E}$, the linear stability analysis can be studied. The following theorem gives the criteria for the local stability of $\hat{E}$ whose proof is similar to theorem 3.3.1 and hence we omit it.

THEOREM 3.4.1 Let the following inequalities hold:

$$\frac{r_{BO}\hat{B}}{K_B(\hat{I},\hat{P})} + \frac{r_{10}\hat{F}_1}{\hat{B}} > \beta\hat{I} + \hat{P}\frac{\partial r_P(\hat{I},\hat{B})}{\partial B} + \frac{r_1}{K_{BO}} \tag{3.35a}$$

$$\gamma_1\hat{I} > \alpha\hat{B} - \frac{r_{BO}\hat{B}^2}{K_B^2(\hat{I},\hat{P})}\frac{\partial K_B(\hat{I},\hat{P})}{\partial I} + \hat{P}\frac{\partial r_P(\hat{I},\hat{B})}{\partial I} + r_2 \tag{3.35b}$$

$$\frac{r_{PO}\hat{P}}{L} > - \hat{B}\frac{\partial r_B(\hat{P})}{\partial P} - \frac{r_{BO}\hat{B}^2}{K_B^2(\hat{I},\hat{P})}\frac{\partial K_B(\hat{I},\hat{P})}{\partial P} + r_3 \tag{3.35c}$$

$$\nu_1 > r_{10} \tag{3.35d}$$

$$\nu_2 > r_{20}\hat{I} \tag{3.35e}$$

$$\nu_3 > r_{30}\hat{P} \tag{3.35f}$$

Then $\hat{E}$ is locally asymptotically stable.

In the following lemma the region of attraction for the system (3.28) is established.

LEMMA 3.4.1 The set

$$\hat{R} = \left\{(B,I,P,F_1,F_2,F_3): 0 \le B \le K_a, \ 0 \le I \le I_a, \ 0 \le P \le P_a,\right.$$

$$\left.0 \le F_1 \le r_1/\nu_1, \ 0 \le F_2 \le r_2 I_a/\nu_2, \ 0 \le F_3 \le r_3 P_a/\nu_3\right\} \text{ attracts}$$

all solutions initiating in the interior of the positive orthant, where

$$K_a = \frac{K_{BO}}{2}\left[1 + \left\{1 + \frac{4r_1 r_{10}}{\nu_1 r_{BO} K_{BO}}\right\}^{1/2}\right], \ I_a = (\beta K_a - \gamma_0)\gamma_1,$$

$$P_a = L r_{PO}(I_a,K_a)/r_{PO}, \ K_a > \gamma_0/\beta.$$

Proof: First we note that

$$\frac{dF_1}{dt} = r_1(1 - \frac{B}{K_{BO}}) - \nu_1 F_1$$

$$\leq r_1 - \nu_1 F_1$$

This implies $F_1 \leq \frac{r_1}{\nu_1} - e^{-\nu_1 t} (\frac{r_1}{\nu_1} - F_1(0))$

Thus if $F_1(0) \leq \frac{r_1}{\nu_1}$, we get $F_1(t) \leq \frac{r_1}{\nu_1}$ for all $t \geq 0$.

Now we have

$$\frac{dB}{dt} = r_B(P)B - \frac{r_{BO} B^2}{K_B(I,P)} - \alpha IB + r_{10}F_1$$

$$\leq r_{BO}B - \frac{r_{BO}B^2}{K_{BO}} + \frac{r_1 r_{10}}{\nu_1}$$

$$= \frac{du}{dt}$$

where $\frac{du}{dt} = r_{BO}B - \frac{r_{BO}B^2}{K_{BO}} + \frac{r_1 r_{10}}{\nu_1}$

We note that the solution of the above equation for $u(0) > 0$ is such that

$$\lim_{t \to \infty} u(t) = K_a = \frac{K_{BO}}{2} \left[1 + \left\{ 1 + \frac{4 r_1 r_{10}}{\nu_1 r_{BO} K_{BO}} \right\}^{1/2} \right]$$

This implies that $\lim_{t \to \infty} \sup B(t) \leq K_a$ for $B(0) \leq K_a$

Further as in lemma 3.3.1,

$$\lim_{t \to \infty} I(t) \leq I_a, \ \lim_{t \to \infty} P(t) \leq P_a, \ \lim_{t \to \infty} F_2(t) \leq r_2 I_a / \nu_2,$$

$$\lim_{t \to \infty} F_3(t) \leq r_3 P_a / \nu_3$$

proving the lemma.

In the following theorem we are able to find sufficient conditions for the global stability of $\hat{E}$.

THEOREM 3.4.2 In addition to the assumptions (3.2) — (3.4), let $r_B(P)$, $K_B(I,P)$, $r_P(I,B)$ satisfy in $\hat{\mathbb{R}}$

$$\hat{K}_m \leq K_B(I,P) \leq K_{BO} \ , \qquad 0 \leq -\frac{\partial r_B(P)}{\partial P} \leq \hat{\rho},$$

$$0 \leq \frac{\partial r_P(I,B)}{\partial I} \leq \hat{\rho}_1 \ , \qquad 0 \leq \frac{\partial r_P(I,B)}{\partial B} \leq \hat{\rho}_2 \ , \qquad (3.36)$$

$$0 \leq -\frac{\partial K_B(I,P)}{\partial I} \leq \hat{k}_1 \ , \qquad 0 \leq -\frac{\partial K_B(I,P)}{\partial P} \leq \hat{k}_2 \ ,$$

for some positive constants $\hat{K}_m$, $\hat{\rho}$, $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{k}_1$, $\hat{k}_2$. Then if the following inequalities hold

$$\left[\frac{r_{BO} \ K_a \hat{k}_1}{\hat{K}_m^2} + \alpha + \beta \right]^2 < \frac{r_{BO} \ \gamma_1}{K_B(\hat{I},\hat{P})} \qquad (3.37a)$$

$$\left[\frac{r_{BO} \ K_a \hat{k}_2}{\hat{K}_m^2} + \hat{\rho} + \hat{\rho}_2 \right]^2 < \frac{r_{BO} \ r_{PO}}{LK_B(\hat{I},\hat{P})} \qquad (3.37b)$$

$$\hat{\rho}_1^2 < \frac{r_{PO} \ \gamma_1}{L} \qquad (3.37c)$$

$\hat{E}$ is globally asymptotically stable with respect to all solutions initiating in the positive octant.

Proof : Taking the following positive definite function about $\hat{E}$,

$$W = [B - \hat{B} - \hat{B} \ \ln(B/\hat{B})] + [I - \hat{I} - \hat{I} \ \ln(I/\hat{I})]$$

$$+ [P - \hat{P} - \hat{P} \ \ln(P/\hat{P})] + \frac{r_{10} K_{BO}}{2r_1 \hat{B}} (F_1 - \hat{F}_1)^2 + \frac{r_{20}}{2r_2} (F_2 - \hat{F}_2)^2$$

$$+ \frac{r_{30}}{2r_3} (F_3 - \hat{F}_3)^2 \qquad (3.38)$$

one can see that the time derivative of W along the solution of (3.28) under the conditions (3.37) is negative definite. By using Liapunov's theorem of stability (La Salle and Lefschetz, 1961) we

conclude that W is a Liapunov function with respect to $\hat{E}$ whose domain contains the region of attraction $\hat{R}$, proving the theorem.

The above theorem implies that if suitable measures are taken to conserve the resource biomass by reforestation and by controlling the undesired level of industrialization and population, an appropriate level of biomass can be maintained.

3.5 EXAMPLES

In this section we give two examples to explain the applicability of the results discussed above by choosing the following functions in model (3.1).

$$r_B(P) = r_{B0} - r_{B1}P$$

$$K_B(I,P) = K_{B0} - K_{B1}I - K_{B2}P \qquad (3.39)$$

$$r_P(I,B) = r_{P0} + r_{P1}I + r_{P2}B$$

where the coefficients are positive.

Example 1.   In this example we choose the following set of parameters in model (3.1).

$\gamma_0 = 0.2$, $\gamma_1 = 4.0$, $\alpha = 0.1$, $\beta = 0.3$, L = 2.0, $r_{B0} = 4.0$,

$r_{B1} = 0.03$, $K_{B0} = 37.1182$, $K_{B1} = 0.04$, $K_{B2} = 0.05$, $r_{P0} = 8.0$,

$r_{P1} = 0.01$, $r_{P2} = 0.02$.

Then it can be checked that the interior equilibrium $E^*(B^*,I^*,P^*)$ of the model (3.1) exists and is given by

$$B^* \simeq 34.0, \quad I^* \simeq 2.5, \quad P^* \simeq 2.1763 .$$

It can also be checked that the conditions (3.19) in theorem 3.3.1 are satisfied and hence E^* is locally asymptotically stable.

By choosing $K_m = 5$ in theorem 3.3.2 and with the above set of parameters and functions, it can further be checked that

conditions (3.21) are satisfied. This shows that E^* is globally asymptotically stable.

Example 2. In this example we choose the same functions given by (3.39) and the following set of parameters in model (3.28).

$\gamma_0 = 0.2$, $\gamma_1 = 4.0$, $\alpha = 0.13$, $\beta = 0.3$, L = 2.0, $I_c = 1.0$,

$P_c = 1.0$, $r_1 = 0.4$, $r_2 = 0.08$, $r_3 = 0.1$, $\nu_1 = 0.15$, $\nu_2 = 0.7$,

$\nu_3 = 0.8$, $r_{10} = 0.12$, $r_{20} = 0.11$, $r_{30} = 0.09$, $r_{B0} = 11.4575$,

$r_{B1} = 0.03$, $K_{B0} = 37.1182$, $K_{B1} = 0.04$, $K_{B2} = 0.05$, $r_{P0} = 8.0$,

$r_{P1} = 0.01$, $r_{P2} = 0.02$.

With the above set of parameters it can be checked that the interior equilibrium $\hat{E}(\hat{B}, \hat{I}, \hat{P}, \hat{F}_1, \hat{F}_2, \hat{F}_3)$ of model (3.28) exists and is given by

$\hat{B} \simeq 35.6$, $\hat{I} \simeq 2.6149$, $\hat{P} \simeq 2.1812$, $\hat{F}_1 \simeq 0.1019$,

$\hat{F}_2 \simeq 0.0703$, $\hat{F}_3 \simeq 0.0227$.

It can also be checked that the conditions (3.35) in theorem 3.4.1 are satisfied and hence $\hat{E}$ is locally asymptotically stable.

By choosing $\hat{K}_m = 5$ in theorem 3.4.2, it can further be checked that the conditions (3.37) in this theorem are also satisfied and hence $\hat{E}$ is globally asymptotically stable.

3.6 CONCLUSIONS

In this chapter, a dynamic model for the depletion of forestry resource caused by industrialization and population pressure is proposed and analysed. It is assumed that growth rate of the forest biomass decreases with population density only while its carrying capacity decreases with both the densities of industrialization and population. It is assumed that the growth

rate of industrialization depends completely on the resource biomass and the growth rate of population increases with the densities of the resource biomass and industrialization. It is further assumed that the resource biomass and population follow generalized logistic equation while the industrialization follows the predator type equation. By analysing the model it is shown that the steady state level of resource biomass decreases as the densities of industrialization and population increase and the magnitude of this level is lower than the case when only either the effect of industrialization or population is taken in account on the resource. It is shown that if the industrialization and population pressures continue without control, the forest biomass may not last long. However, if suitable steps of conservation are taken, keeping in view the sustainable use of resource, an appropriate level of resource biomass can be maintained.
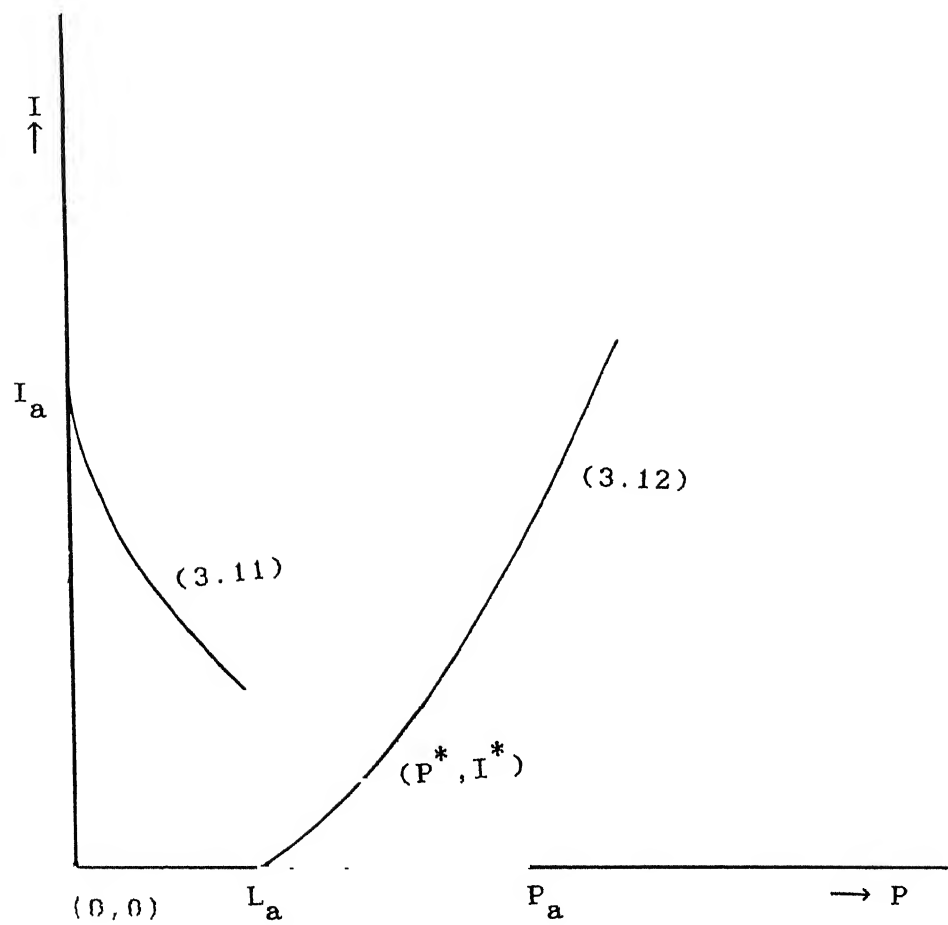
Fig 3.1
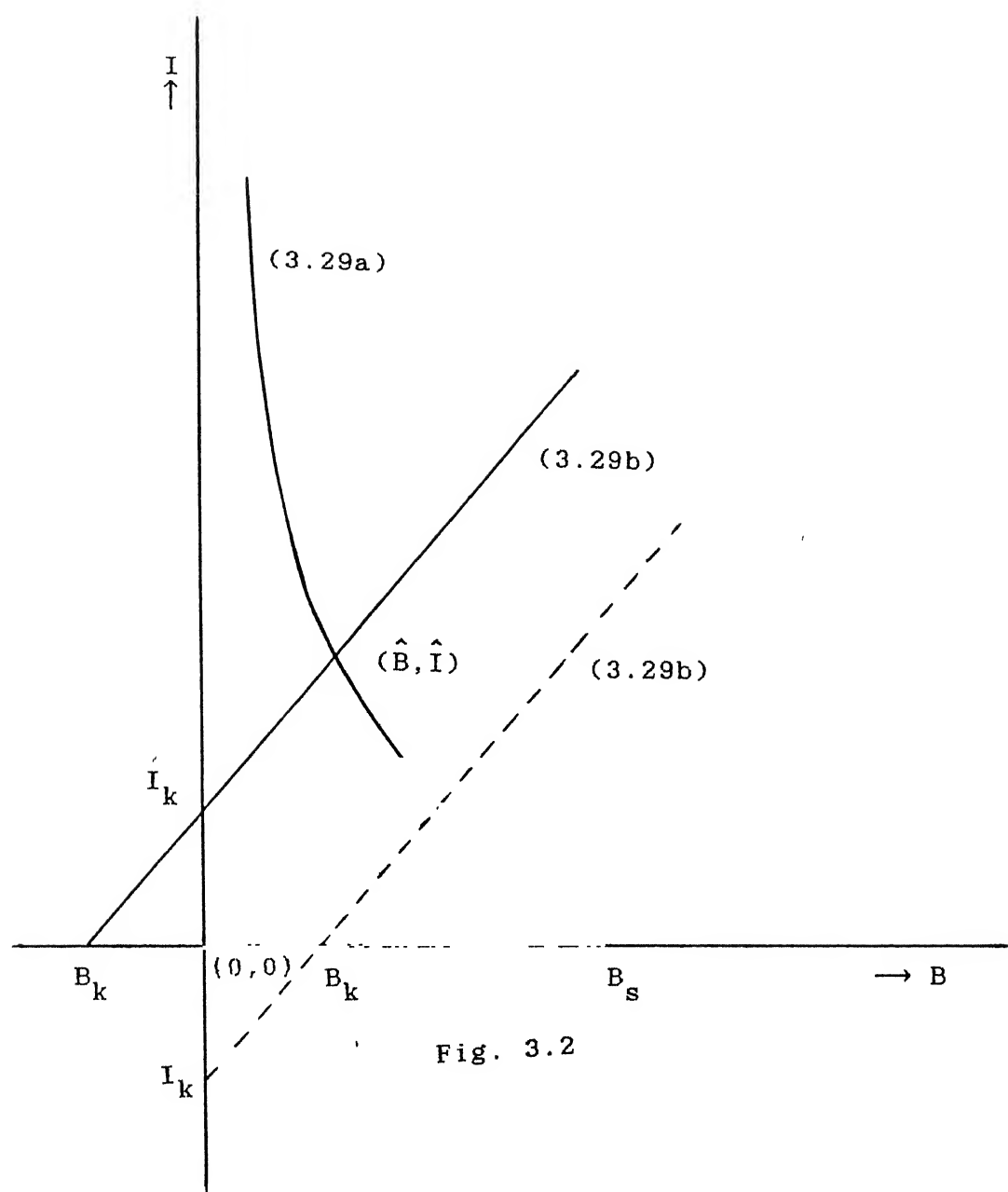
Fig. 3.2

CHAPTER IV

MODELLING THE DEPLETION AND CONSERVATION OF FORESTRY RESOURCES: EFFECTS OF POPULATION AND POLLUTION

4.0 INTRODUCTION

One of the challenging problems which society faces today is the depletion of forestry resources causing climate change, drought, extinction of species, etc. affecting the biodiversity due to increasing population and pollution particularly in the third world countries. The depletion of forestry resources (the term resource is used for all forestry resources such as tree/plant population, wildlife population, etc.) are caused by clearance of forest land for agriculture, resettlement and colonization, cutting of trees for fuel and fodder, etc. required to meet the need of increased population. Industrial uses of forest land for the wood and paper industries and for mining are also causes for forestry resource depletion. A typical example where depletion of forest has taken place in recent times is the Doon Valley, located in the northern part of Uttar Pradesh, India. According to a report prepared under the auspices of International Institute for Applied Systems Analysis, the main reasons for the depletion of forest biomass and the threat to ecological stability in this Valley are growth of human and livestock populations, limestone quarrying, wood based industries and various kinds of industrial discharges and chemical spills in the forms of smokes, poisonous gas fumes, hazardous wastes polluting the air and affecting the forest biomass, Munn and Fedorov (1986), Shukla et al. (1989). In particular, Shukla et al. (1989) studied the effect

of population on the depletion of forestry resource and noted that increased population may lead to the extinction of the resource.

In recent years some investigations have been conducted to study the isolated effect of population or pollutant(toxicant) on a biological species using mathematical models, Hallam et al. (1983a,b), Hallam and De Luna (1984), De Luna and Hallam (1987), Shukla et al. (1989), Freedman and Shukla (1991), Huaping and Ma (1991). But in these studies the depletion of forestry resource due to the combined effect of population and pollution have not been studied.

In this chapter we therefore propose a mathematical model to study the depletion of forestry resources due to an increase in both population and pollution in the habitat under consideration. It is assumed that the biomass and population densities are governed by generalized logistic equations and the growth rate of population increases with the resource biomass density, but the carrying capacity remains independent of it. It is further assumed that the pollutant is emitted into the environment with a prescribed rate and is depleted by some natural degradation factors. It is considered further that the growth rate of the resource biomass decreases with the increase in population density and with the uptake concentration of the pollutant, but its carrying capacity decreases with the increase in population density and concentration of the pollutant in the environment. Stability theory (La Salle and Lefschetz (1961)) is used for the model analysis to study the depletion of resource biomass caused by the increase in population and pollution emitted into the environment with instantaneous, constant, population dependent or

periodic rates.

A model for conservation of resource biomass by controlling the population and pollution is also proposed and analysed. It is assumed here that the effort applied to conserve the resource biomass is proportional to the depleted level of the biomass from its carrying capacity. It is also assumed that the efforts applied to control the population and pollution are proportional to the undesired level densities of population and pollution respectively.

We assume that all functions utilized here are sufficiently smooth that solutions to initial value problems exist uniquely and are continuable for all positive time.

4.1 MATHEMATICAL MODEL

We consider a forest habitat where we wish to model the depletion of forest biomass due to increases in both population and pollution in the environment. Keeping in view the discussions of the problem above, the differential equations governing the system can be written as follows, (Shukla et al. (1989), Freedman and Shukla (1991)).

$$\frac{dB}{dt} = r_B(U,P)B - \frac{r_{BO}B^2}{K_B(T,P)}$$

$$\frac{dP}{dt} = r(B)P - \frac{r_0 P^2}{L}$$

$$\frac{dT}{dt} = Q - \delta_0 T - \alpha BT + \pi\nu BU$$

$$\frac{dU}{dt} = -\delta_1 U + \alpha BT - \nu BU$$

(4.1)

$B(0) = B_0 \geq 0$, $P(0) = P_0 \geq 0$, $T(0) = T_0 \geq 0$, $U(0) \geq kB_0$,

$0 \leq \pi \leq 1$.

Here B(t) is the density of the resource biomass, P(t) is the density of the population, T(t) is the concentration of the pollutant present in the environment and U(t) is the uptake concentration of pollutant by the resource biomass, Q is the rate of emission of pollutant into the environment which is either zero, constant, population dependent or periodic, $\delta_0 > 0$ and $\delta_1 > 0$ are the natural washout rate coefficients of pollutant present in the environment and resource biomass respectively, α is the depletion rate coefficient of pollutant present in the environment due to its uptake by the resource biomass. Also the uptake concentration may decrease with rate coefficient ν due to decaying of resource biomass and a fraction π of which may reenter into the environment.

In the relation $U(0) \geq kB_0$, $k \geq 0$ is the proportionality constant determining the measure of initial pollutant concentration U(0) in the initial resource biomass density B_0.

For convenience, where there is no confusion, a prime denotes the derivative of a function with respect to its argument.

In writing down the model (4.1) it is assumed that the densities of resource biomass and population follow generalized logistic models. It is further assumed that the growth rate of uptake concentration U(t) increases with the same amount by which the growth rate of environmental pollutant concentration T(t) decreases and is assumed to be proportional to resource biomass density as well as the environmental concentration of the pollutant (i.e. αBT). In case the growth of industrialization is logistic, the second equation of the model (4.1) may be thought of as governing the cumulative effect of population and

industrialization, Quaddus (1985).

In our model (4.1), the function $r_B(U,P)$ denotes the growth rate coefficient of resource biomass which decreases with the uptake concentration of pollutant and with the population density i.e.

$$r_B(0,0) = r_{BO} > 0, \quad \frac{\partial r_B(U,P)}{\partial U} < 0, \quad \frac{\partial r_B(U,P)}{\partial P} < 0 \text{ for } U \geq 0, P \geq 0,$$

and $r_B(0,\bar{\bar{P}}) = r_B(\bar{\bar{U}},0) = 0$ for some $\bar{\bar{P}} > 0, \bar{\bar{U}} > 0$. (4.2)

The function $K_B(T,P)$ denotes the maximum resource biomass density which the environment can support and it decreases with T as well as with P i.e.

$$K_B(0,0) = K_{BO} > 0, \quad \frac{\partial K_B(T,P)}{\partial T} < 0$$

$$\text{and} \quad \frac{\partial K_B(T,P)}{\partial P} < 0 \text{ for } T \geq 0, P \geq 0. \tag{4.3}$$

L denotes the maximum population density which the environment can support and the function $r(B)$ denotes its growth rate coefficient which increases as the density of the resource biomass increases i.e.

$$r(0) = r_0 > 0, \quad r'(B) \geq 0 \quad \text{for } B \geq 0 \tag{4.4}$$

In this chapter, we analyse our model (4.1) for four different cases, namely $Q = 0$, $Q = Q_0$ a positive constant, $Q = Q(P)$ satisfying

$$Q(0) > 0, \quad Q'(P) > 0 \text{ for } P \geq 0 \tag{4.5}$$

or Q is periodic.

4.2 MATHEMATICAL ANALYSIS

CASE I: Q = 0

In this case, our model (4.1) has four nonnegative equilibria, namely $E_0(0,0,0,0)$, $E_1(K_{BO},0,0,0)$, $E_2(0,L,0,0)$, and $E_3(B_1,P_1,0,0)$ where B_1 and P_1 are given by

$$r_B(0,P_1)K_B(0,P_1)/r_{Bo} = B_1 \qquad (4.6a)$$

$$P_1 = r(B_1)L/r_0 \qquad (4.6b)$$

From (4.6a), we note that even in the absence of pollution the resource biomass density decreases as the population density increases and may tend to zero for large population density.

From (4.6a), we also note that B_1 is a decreasing function of P_1 starting from K_{BO} and from (4.6b), we see that P_1 is an increasing function of B starting from L and hence the two isoclines intersect at a unique point (B_1,P_1) iff

$$L < \bar{\bar{P}} \qquad (4.7)$$

The stability analysis of the equilibria can be studied by computing the variational matrices corresponding to each equilibrium.

Let M_1 = variational matrices corresponding to E_i, i = 0, 1, 2, 3. Then we have

$$M_0 = \begin{bmatrix} r_{BO} & 0 & 0 & 0 \\ 0 & r_0 & 0 & 0 \\ 0 & 0 & -\delta_0 & 0 \\ 0 & 0 & 0 & -\delta_1 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} -r_{BO} & G_2 & r_{BO}\dfrac{\partial K_B(0,0)}{\partial T} & K_{BO}\dfrac{\partial r_B(0,0)}{\partial U} \\[2mm] 0 & r(K_{BO}) & 0 & 0 \\[2mm] 0 & 0 & -\delta_0 - \alpha K_{BO} & \pi\nu K_{BO} \\[2mm] 0 & 0 & \alpha K_{BO} & -\delta_1 - \nu K_{BO} \end{bmatrix}$$

where

$$G_2 = K_{BO}\frac{\partial r_B(0,0)}{\partial P} + r_{BO}\frac{\partial K_B(0,0)}{\partial P} < 0 \qquad (4.8a)$$

$$M_2 = \begin{bmatrix} r_{BO}(0,L) & 0 & 0 & 0 \\[2mm] r'(0)L & -r_0 & 0 & 0 \\[2mm] 0 & 0 & -\delta_0 & 0 \\[2mm] 0 & 0 & 0 & -\delta_1 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} -r_B(0,P_1) & G_3 & \dfrac{r_B^2(0,P_1)}{r_{BO}}\dfrac{\partial K_B(0,P_1)}{\partial T} & B_1\dfrac{\partial r_B(0,P_1)}{\partial U} \\[2mm] r'(B_1)P_1 & -r(B_1) & 0 & 0 \\[2mm] 0 & 0 & -\delta_0 - \alpha B_1 & \pi\nu B_1 \\[2mm] 0 & 0 & \alpha B_1 & -\delta_1 - \nu B_1 \end{bmatrix}$$

where

$$G_3 = \frac{r_B^2(0,P_1)}{r_{BO}}\frac{\partial K_B(0,P_1)}{\partial P} + B_1\frac{\partial r_B(0,P_1)}{\partial P} < 0 \qquad (4.8b)$$

From M_0, we note that E_0 is a saddle point whose stable manifold is locally in the T-U space and whose unstable manifold is locally in the B-P space. From M_1, we note that E_1 is also a saddle point with stable manifold locally in B-T-U space and

unstable manifold locally in the P-direction. From M_2, we note that E_2 is also a saddle point with stable manifold locally in P-T-U space and unstable manifold locally in the B-direction. From M_3, we can check that E_3 is locally asymptotically stable in B-P-T-U space.

We now show that E_3 is also globally asymptotically stable.

THEOREM 4.2.1   If $B(0) > 0$, $P(0) > 0$, then E_3 is globally asymptotically stable in the region

$$A = \left\{ (B,P,T,U) : 0 \leq B \leq K_{B0},\ 0 \leq P \leq P_m,\ T = 0,\ U = 0,\ \text{where } P_m = r(K_{B0})L/r_0 \right\}.$$

Proof: From (4.1) we have

$$\frac{dB}{dt} = r_B(U,P)B - \frac{r_{B0}B^2}{K_B(T,P)}$$

$$\leq r_{B0}B - r_{B0}B^2/K_{Bo}$$

Hence $\lim_{t \to \infty} B(t) \leq K_{B0}$

Also we have

$$\frac{dP}{dt} = r(B)P - \frac{r_0 P^2}{L}$$

$$\leq r(K_{B0})P - \frac{r_0 P^2}{L}$$

and hence $\lim_{t \to \infty} P(t) \leq P_m$

Again we have

$$\frac{dT}{dt} + \frac{dU}{dt} = -\delta_0 T - \delta_1 U - (1 - \pi)\nu BU$$

$$\leq -\delta(T + U), \quad \delta = \min(\delta_0, \delta_1).$$

This implies that $T(t) + U(t) \leq (T(0) + U(0))e^{-\delta t}$

and hence $\lim_{t \to \infty} T(t) = \lim_{t \to \infty} U(t) = 0$.

This shows that the system is dissipative.

Since $B(0) > 0$ and $P(0) > 0$, the theorem follows.

This theorem shows that in the case of instantaneous spill, the pollutant may be washed out completely and the resource biomass would then settle down to a lower equilibrium than its carrying capacity whose magnitude will depend upon the density of the equilibrium level of the population. It should be noted here that even in the absence of pollutant the resource biomass may tend to extinction if the population increases without limit.

CASE II: $Q = Q_0$ (a positive constant)

In this case, the model (4.1) has four nonnegative equilibria, viz $E_1^*(0,0,Q_0/\delta_0,0)$, $E_2^*(0,L,Q_0/\delta_0,0)$, $\tilde{E}(\tilde{B},0,\tilde{T},\tilde{U})$ and $E^*(B^*,P^*,T^*,U^*)$.

We will show the existence of $\tilde{E}(\tilde{B},0,\tilde{T},\tilde{U})$ as follows :

Here $\tilde{B}$, $\tilde{T}$, $\tilde{U}$ are the positive solution of the system of algebraic equations

$$B = r_B(U,0)K_B(T,0)/r_{Bo} \tag{4.9a}$$

$$T = \frac{Q_0 + \pi\nu BU}{\delta_0 + \alpha B} \tag{4.9b}$$

$$U = \frac{\alpha BT}{\delta_1 + \nu B} \tag{4.9c}$$

Substituting T from (4.9b) in (4.9c), we get

$$U = \frac{\alpha Q_0 B}{f(B)} = h(B) \quad ,(\text{say}) \tag{4.10}$$

where

$$f(B) = \delta_0\delta_1 + (\alpha\delta_1 + \nu\delta_0)B + (1 - \pi)\nu\alpha B^2 \tag{4.11}$$

We note that U increases as Q_0 increases.

Substituting U from (4.10) in (4.9a), we get

$$T = g(B) \tag{4.12}$$

where

$$g(B) = \frac{Q_0 + \pi \nu B \ h(B)}{\delta_0 + \alpha B} \tag{4.13}$$

which increases as Q_0 increases.

Substituting T from (4.12) in (4.9a), we get

$$r_{B0} B = r_B(U,0) K_B(g(B),0) \tag{4.14}$$

To show the existence of $\tilde{E}$, it suffices to show that the isoclines (4.10) and (4.14) intersect at a unique point.

From (4.10) we note the following :

When $B \to 0$, $U \to 0$ \hfill (4.15a)

When $B \to K_{B0}$, $U \to \dfrac{\alpha Q_0 K_{B0}}{f(K_{B0})} > 0$ \hfill (4.15b)

We also have

$$\frac{dU}{dB} = \frac{\alpha Q_0}{f^2(B)} \ [\delta_0 \delta_1 - (1 - \pi)\nu \alpha B^2] \tag{4.15c}$$

$$\lim_{B \to 0} \frac{dU}{dB} = \frac{\alpha Q_0}{\delta_0 \delta_1} > 0 \tag{4.15d}$$

We note that $\dfrac{dU}{dB} > 0$ for all B iff

$$\delta_0 \delta_1 > (1 - \pi)\nu \alpha B^2 \tag{4.16}$$

From (4.14) we note the following :

When $B \to 0$, $U \to \bar{U}$ \hfill (4.17a)

When $U \to 0$, $B \to B_c$ \hfill (4.17b)

where B_c is given by

$$B_c = K_B(g(B_c),0) \tag{4.17c}$$

such that $B_c \le K_{B0}$

Further we have

$$K_B(g(B),0) \frac{\partial r_B}{\partial U} \frac{dU}{dB} = r_{BO} - r_B(U,0) \frac{\partial K_B}{\partial g} \frac{dg}{dB} \qquad (4.17d)$$

This shows that $\frac{dU}{dB} < 0$ iff

$$r_B(U,0) \frac{\partial K_B}{\partial g} \frac{dg}{dB} < r_{BO} \qquad (4.18)$$

Thus the isocline (4.10) is an increasing function of B starting from zero under the condition (4.16) and the isocline (4.14) is a decreasing function of U starting from $\bar{U}$ under the condition (4.18). Hence the two isoclines (4.10) and (4.14) must intersect at a unique point $(\tilde{B}, \tilde{U})$. After knowing the value of $\tilde{B}$ and $\tilde{U}$, $\tilde{T}$ can be computed from (4.9b).

Existence of $E^*(B^*, P^*, T^*, U^*)$:

Here B^*, P^*, T^*, U^* are positive solutions of the following system of algebraic equations

$$B = r_B(h(B),P)K_B(g(B),P)/r_{BO} \qquad (4.19a)$$

$$P = r(B)L/r_0 \qquad (4.19b)$$

$$T = g(B) \qquad (4.19c)$$

$$U = h(B) \qquad (4.19d)$$

where

$$g(B) = \frac{Q_0 + \pi \nu B\ h(B)}{\delta_0 + \alpha B} \qquad (4.20a)$$

$$h(B) = \alpha Q_0 B / [\delta_0 \delta_1 + (\alpha \delta_1 + \nu \delta_0)B + (1 - \pi)\nu \alpha B^2] \qquad (4.20b)$$

From (4.19a) we note the following :

When $B \rightarrow 0$, $P \rightarrow \bar{\bar{P}}$ \qquad (4.21a)

When $P \rightarrow 0$, $B \rightarrow B_e$ \qquad (4.21b)

where B_e is a positive solution of

$$F(B) = r_{B0}B - r_B(h(B),0)K_B(g(B),0) = 0 \qquad (4.22)$$

The existence of B_e in the interval $0 < B_e < K_{B0}$ is obvious, since

$$F(0) = - r_{B0}K_B(Q_0/\delta_0,0) < 0 \qquad (4.23a)$$

and $\quad F(K_{B0}) > 0 \qquad (4.23b)$

Also we have

$$\frac{dP}{dB}\left[r_B(h(B),P) \frac{\partial K_B}{\partial P} + K_B(g(B),P) \frac{\partial r_B}{\partial P} \right] =$$

$$r_{B0} - r_B(h(B),P) \frac{\partial K_B}{\partial g} \frac{dg}{dB} - K_B(g(B),P) \frac{\partial r_B}{\partial h} \frac{dh}{dB} \qquad (4.24)$$

This shows that $\frac{dP}{dB} < 0$ iff

$$r_B(h(B),P) \frac{\partial K_B}{\partial g} \frac{dg}{dB} - K_B(g(B),P) \frac{\partial r_B}{\partial h} \frac{dh}{dB} < r_{B0} \qquad (4.25)$$

From the above analysis it is clear that the isocline (4.19a) is a decreasing function of B starting from $\bar{\bar{P}}$ under the condition (4.25). From (4.19b) it is obvious that this isocline is an increasing function of B starting from L. Hence the two isoclines (4.19a) and (4.19b) intersect at a unique point [see fig.4.1] iff the inequality (4.7) is satisfied. The intersection value of the above two isoclines gives the B^*-P^* coordinates of E^*. Knowing the value of B^* and P^*, then T^* and U^* can be computed from (4.19c) and (4.19d) respectively.

Now using the general notation for variational matrices (i.e. M_1^* is the variational matrix corresponding to E_1^*), we have

$$
M_1^* = \begin{bmatrix}
r_{BO} & 0 & 0 & 0 \\
0 & r_0 & 0 & 0 \\
-\dfrac{\alpha Q_0}{\delta_0} & 0 & -\delta_0 & 0 \\
\dfrac{\alpha Q_0}{\delta_0} & 0 & 0 & -\delta_1
\end{bmatrix}
$$

$$
M_2^* = \begin{bmatrix}
r_B(0,L) & 0 & 0 & 0 \\
r'(0)K & -r_0 & 0 & 0 \\
-\dfrac{\alpha Q_0}{\delta_0} & 0 & -\delta_0 & 0 \\
\dfrac{\alpha Q_0}{\delta_0} & 0 & 0 & -\delta_1
\end{bmatrix}
$$

$$
\tilde{M} = \begin{bmatrix}
-r_B(\tilde{U},0) & \tilde{G} & \dfrac{r_B^2(\tilde{U},0)}{r_{BO}}\dfrac{\partial K_B(\tilde{T},0)}{\partial T} & \tilde{B}\dfrac{\partial r_B(\tilde{U},0)}{\partial U} \\
0 & r(\tilde{B}) & 0 & 0 \\
-\alpha\tilde{T} + \pi\nu\tilde{U} & 0 & -\delta_0 - \alpha\tilde{B} & \pi\nu\tilde{B} \\
\alpha\tilde{T} - \nu\tilde{U} & 0 & \alpha\tilde{B} & -\delta_1 - \nu\tilde{B}
\end{bmatrix}
$$

where

$$
\tilde{G} = \frac{r_B^2(\tilde{U},0)}{r_{BO}}\frac{\partial K_B(\tilde{T},0)}{\partial P} + \tilde{B}\frac{\partial r_B(\tilde{U},0)}{\partial P} < 0 \tag{4.26}
$$

$$M^* = \begin{bmatrix} -r_B(U^*,P^*) & G^* & \dfrac{r_B^2(U^*,P^*)}{r_{Bo}}\dfrac{\partial K_B(T^*,P^*)}{\partial T} & B^*\dfrac{\partial r_B(U^*,P^*)}{\partial U} \\[2ex] r'(B^*)P^* & -r(B^*) & 0 & 0 \\[2ex] -\alpha T^* + \pi\nu U^* & 0 & -\delta_o - \alpha B^* & \pi\nu B^* \\[2ex] \alpha T^* - \nu U^* & 0 & \alpha B^* & -\delta_1 - \nu B^* \end{bmatrix}$$

where

$$G^* = \frac{r_B^2(U^*,P^*)}{r_{BO}}\frac{\partial K_B(T^*,P^*)}{\partial P} + B^*\frac{\partial r_B(U^*,P^*)}{\partial P} < 0 \qquad (4.27)$$

From M_1^*, we note that E_1^* is a saddle point whose stable manifold is locally in the T-U space and whose unstable manifold is locally in the B-P space. From M_2^*, we note that E_2^* is also a saddle point with stable manifold locally in P-T-U space and unstable manifold locally in the B-direction. From $\tilde{M}$, we note that $\tilde{E}$ is unstable in the P-direction. The local stability behavior of $\tilde{E}$ in B-T-U space can be checked as in Freedman and Shukla (1991) and is similar to E^*, the conditions for the local stability of which is given in the following theorem.

THEOREM 4.2.2 Let the following inequalities hold

$$r_B(U^*,P^*) > 2\alpha T^* + r'(B^*)P^* - (1 + \pi)\nu U^* \qquad (4.28a)$$

$$r(B^*) > -G^* \qquad (4.28b)$$

$$\delta_0 > -\frac{r_B^2(U^*,P^*)}{r_{Bo}}\frac{\partial K_B(T^*,P^*)}{\partial T} \qquad (4.28c)$$

$$\delta_1 > (\pi - 1)\nu B^* - B^*\frac{\partial r_B(U^*,P^*)}{\partial U} \qquad (4.28d)$$

Then E^* is locally asymptotically stable.

Proof: If inequalities (4.28) hold, then by Gershgorin's theorem (Lancaster and Tismanetsky (1985), p.371), all eigen values of M^* have negative real parts, and the theorem follows.

In the following theorem, we are able to write down conditions which guarantee that E^* is globally asymptotically stable. To prove this theorem, we require a lemma which establishes the region of attraction for the system (4.1). The ideas used here are developed in Hsu (1978) and Freedman (1987).

LEMMA 4.2.1   The set

$$\mathbb{R} = \left\{ (B,P,T,U): \ 0 \leq B \leq K_{BO}, \ 0 \leq P \leq P_m, \ 0 \leq T + U \leq Q_0/\delta, \right.$$

$$\left. \delta = \min (\delta_0, \ \delta_1), \ P_m = r(K_{BO})L/r_0 \right\} \text{ is a region of attraction}$$

for all solutions initiating in the positive octant.

Proof: As before, $\lim_{t \to \infty} B(t) \leq K_{BO}$

$$\lim_{t \to \infty} P(t) \leq P_m$$

$$\text{and } \frac{dT}{dt} + \frac{dU}{dt} \leq - \delta(T + U) + Q_0$$

Hence $\lim_{t \to \infty} [T(t) + U(t)] \leq Q_0/\delta$, proving the lemma.

THEOREM 4.2.3   In addition to the assumptions (4.2)—(4.4), let $r(B)$, $r_B(U,P)$, $K_B(T,P)$ satisfy in $\mathbb{R}$

$$0 \leq r'(B) \leq \rho, \quad K_c \leq K_B(T,P) \leq K_{BO},$$

$$0 \leq - \frac{\partial r_B(U,P)}{\partial U} \leq \rho_1, \quad 0 \leq - \frac{\partial r_B(U,P)}{\partial P} \leq \rho_2, \quad (4.29)$$

$$0 \leq - \frac{\partial K_B(T,P)}{\partial T} \leq k_1, \quad 0 \leq - \frac{\partial K_B(T,P)}{\partial P} \leq k_2,$$

for some positive constants ρ, ρ_1, ρ_2, k_1, k_2, K_c. Then if the following inequalities hold

$$\left[\rho + \rho_2 + \frac{r_{BO} K_{BO} k_2}{K_c^2} \right]^2 < \frac{4 r_O r_{BO}}{3I. \, K_B(T^*, P^*)} \qquad (4.30a)$$

$$\left[\frac{r_{BO} K_{BO} k_1}{K_c^2} + \frac{\alpha Q_0}{\delta} + \pi \nu U^* \right]^2 < \frac{2 r_{BO}}{3 K_B(T^*, P^*)} (\delta_0 + \alpha B^*) \qquad (4.30b)$$

$$\left[\rho_1 + \frac{\nu Q_0}{\delta} + \alpha T^* \right]^2 < \frac{2 r_{BO}}{3 K_B(T^*, P^*)} (\delta_1 + \nu B^*) \qquad (4.30c)$$

$$\left[\pi \nu + \alpha \right]^2 K_{BO}^2 < (\delta_0 + \alpha B^*)(\delta_1 + \nu B^*) \qquad (4.30d)$$

E^* is globally asymptotically stable with respect to all solutions initiating in the positive octant.

Proof: We consider the following positive definite function about E^*,

$$V(B,P,T,U) = (B - B^* - B^* \ln \frac{B}{B^*}) + (P - P^* - P^* \ln \frac{P}{P^*})$$

$$+ \frac{1}{2} (T - T^*)^2 + \frac{1}{2} (U - U^*)^2 \qquad (4.31)$$

Differentiating V with respect to t along the solutions of (4.1) and using (4.19) we get after some algebraic manipulations

$$\frac{dV}{dt} = - \frac{r_{BO}}{K_B(T^*, P^*)} (B - B^*)^2 - \frac{r_0}{L} (P - P^*)^2$$

$$- (\delta_0 + \alpha B^*)(T - T^*)^2 - (\delta_1 + \nu B^*)(U - U^*)^2$$

$$+ (B - B^*)(P - P^*)\left[\eta(B) + \eta_2(U^*, P) - r_{BO} B \xi_2(T^*, P) \right]$$

$$+ (B - B^*)(T - T^*)\left[- r_{BO} B \xi_1(T, P) - \alpha T + \pi \nu U^* \right]$$

$$+ (B - B^*)(U - U^*)\left[\eta_1(U, P) - \nu U + \alpha T^* \right]$$

$$+ (T - T^*)(U - U^*)\left[\pi \nu B + \alpha B \right] \qquad (4.32)$$

where

$$\eta(B) = \begin{cases} [r(B) - r(B^*)]/(B - B^*), & B \neq B^* \\ r'(B^*), & B = B^* \end{cases} \qquad (4.33a)$$

$$\eta_1(U,P) = \begin{cases} \dfrac{r_B(U,P) - r_B(U^*,P)}{U - U^*}, & U \neq U^* \\[2mm] \dfrac{\partial r_B(U^*,P)}{\partial U}, & U = U^* \end{cases} \qquad (4.33b)$$

$$\eta_2(U^*,P) = \begin{cases} \dfrac{r_B(U^*,P) - r_B(U^*,P^*)}{P - P^*}, & P \neq P^* \\[2mm] \dfrac{\partial r_B(U^*,P^*)}{\partial P}, & P = P^* \end{cases} \qquad (4.33c)$$

$$\xi_1(T,P) = \begin{cases} \left[\dfrac{1}{K_B(T,P)} - \dfrac{1}{K_B(T^*,P)}\right]/(T - T^*), & T \neq T^* \\[3mm] -\dfrac{1}{K_B^2(T^*,P)}\dfrac{\partial K_B(T^*,P)}{\partial T}, & T = T^* \end{cases} \qquad (4.33d)$$

$$\xi_2(T^*,P) = \begin{cases} \left[\dfrac{1}{K_B(T^*,P)} - \dfrac{1}{K_B(T^*,P^*)}\right]/(P - P^*), & P \neq P^* \\[3mm] -\dfrac{1}{K_B^2(T^*,P^*)}\dfrac{\partial K_B(T^*,P^*)}{\partial P}, & P = P^* \end{cases} \qquad (4.33e)$$

From (4.31) and the mean value theorem, we note that

$$|\eta(B)| \leq \rho, \quad |\eta_1(U,P)| \leq \rho_1, \quad |\eta_2(U^*,P)| \leq \rho_2 \qquad (4.34)$$

$$|\xi_1(T,P)| \leq k_1/K_c^2, \quad |\xi_2(T^*,P)| \leq k_2/K_c^2.$$

Now $\dfrac{dV}{dt}$ can further be written as the sum of the quadratics

$$\frac{dV}{dt} = -\frac{1}{2}a_{11}(B - B^*)^2 + a_{12}(B - B^*)(P - P^*) - \frac{1}{2}a_{22}(P - P^*)^2$$

$$-\frac{1}{2}a_{11}(B - B^*)^2 + a_{13}(B - B^*)(T - T^*) - \frac{1}{2}a_{33}(T - T^*)^2$$

$$-\frac{1}{2}a_{11}(B - B^*)^2 + a_{14}(B - B^*)(U - U^*) - \frac{1}{2}a_{44}(U - U^*)^2$$

$$-\frac{1}{2}a_{33}(T - T^*)^2 + a_{34}(T - T^*)(U - U^*) - \frac{1}{2}a_{44}(U - U^*)^2 \qquad (4.35)$$

where

$$a_{11} = \frac{2r_{B0}}{3K_B(T^*, P^*)}, \quad a_{22} = \frac{2r_0}{L}, \quad a_{33} = \delta_0 + \alpha B^*, \quad a_{44} = \delta_1 + \nu B^*,$$

$$a_{12} = \eta(B) + \eta_2(U^*, P) - r_{B0}B\xi_2(T^*, P),$$

$$a_{13} = -r_{B0}B\xi_1(T, P) - \alpha T + \pi\nu U^*,$$

$$a_{14} = \eta_1(U, P) - \nu U + \alpha T^*, \quad a_{34} = \pi\nu B + \alpha B$$

Then sufficient conditions for $\frac{dV}{dt}$ to be negative definite are that the following inequalities hold.

$$a_{12}^2 < a_{11}a_{22} \qquad (4.36a)$$

$$a_{13}^2 < a_{11}a_{33} \qquad (4.36b)$$

$$a_{14}^2 < a_{11}a_{44} \qquad (4.36c)$$

$$a_{34}^2 < a_{33}a_{44} \qquad (4.36d)$$

We note that (4.30a) ⇒ (4.36a), (4.30b) ⇒ (4.36b), (4.30c) ⇒ (4.36c) and (4.30d) ⇒ (4.36d). Thus V is a Liapunov function with respect to E^* whose domain contains the region ℝ, proving the theorem.

This theorem shows that in the case of constant emission of pollutant (Q = constant) in the environment, the resource biomass may settle down to its equilibrium level which is lower than its

carrying capacity as compared to the case $Q = 0$, the magnitude depending upon the equilibrium level of the population as well as influx and washout rates of the pollutant present in the environment. It is also pointed out here that if the population and pollution continues without control, the density of resource biomass decreases considerably and may tend to zero sooner than the case of no population in the habitat.

CASE III: $Q = Q(P)$

In this case, again the model (4.1) has four nonnegative equilibria, namely $\hat{E}_1(0,0,Q(0)/\delta_0,0)$, $\hat{E}_2(0,L,Q(L)/\delta_0,0)$, $\hat{E}_3(\hat{B}_3,0,\hat{T}_3,\hat{U}_3)$ and $\hat{E}(\hat{B},\hat{P},\hat{T},\hat{U})$. Here $\hat{B}_3$, $\hat{T}_3$, $\hat{U}_3$ can be obtained by (4.9) by replacing Q_0 by $Q(0)$ and the existence of $\hat{E}_3$ can be seen under the similar fashion as that of $\tilde{E}$ in case II. Also $\hat{B}$, $\hat{P}$, $\hat{T}$, $\hat{U}$ can be obtained by (4.19) by replacing Q_0 by $f_1(B)$ and the existence of $\hat{E}$ can be seen in the similar way as that of E^* in case II, where $f_1(B)$ is given by

$$f_1(B) = Q(r(B)L/r_0) \qquad (4.37)$$

Further by computing the variational matrices corresponding to these equilibria, it can be seen that the local stability behavior of $\hat{E}_1$, $\hat{E}_2$ and $\hat{E}_3$ coincides with those of E_1^*, E_2^* and $\tilde{E}$ of case II. The conditions for local stability of $\hat{E}$ are given in the following theorem whose proof is similar to theorem 4.2.2 and hence is omitted.

THEOREM 4.2.4   Let the following inequalities hold

$$r_B(\hat{U},\hat{P}) > 2\alpha\hat{T} + r'(\hat{B})\hat{P} - (1 + \pi)\nu\hat{U} \qquad (4.38a)$$

$$r(\hat{B}) > -\hat{G} + Q'(\hat{P}) \qquad (4.38b)$$

$$\delta_0 \; > \; - \; \frac{r_B^2(\hat{U},\hat{P})}{r_{BO}} \; \frac{\partial K_B(\hat{T},\hat{P})}{\partial T} \qquad (4.38c)$$

$$\delta_1 \; > \; (\pi - 1)\nu\hat{B} \; - \; \hat{B} \; \frac{\partial r_B(\hat{U},\hat{P})}{\partial U} \qquad (4.38d)$$

Then $\hat{E}$ is locally asymptotically stable, where $\hat{G}$ can be obtained by replacing X^* $(X = P, T, U)$ by $\hat{X}$ in (4.27).

In the following theorem we have found certain conditions under which $\hat{E}$ is globally asymptotically stable. To prove this theorem we have to establish a lemma for the region of attraction of our system whose proof is similar to lemma 4.2.1 and hence is omitted.

LEMMA 4.2.2   The set
$$\hat{R} = \left\{ (B,P,T,U): \; 0 \leq B \leq K_{BO}, \; 0 \leq P \leq P_m, \; 0 \leq T + U \leq Q_m/\delta, \right.$$

$$\left. \delta = \min \; (\delta_o, \; \delta_1), \; P_m = r(K_{BO})L/r_0, \; Q_m = Q(P_m) \right\} \text{ is a region}$$

of attraction for all solutions initiating in the positive octant.

The proof of the following theorem is similar to theorem 4.2.3 and hence is omitted.

THEOREM 4.2.5   In addition to the assumptions (4.2)—(4.5), let $r(B)$, $r_B(U,P)$, $K_B(T,P)$, $Q(P)$ satisfy in $\hat{R}$

$$0 \leq r'(B) \leq \hat{\rho}, \; 0 < Q'(P) \leq \hat{q}_0, \; \hat{K}_c \leq K_B(T,P) \leq K_{BO},$$

$$0 \leq - \frac{\partial r_B(U,P)}{\partial U} \leq \hat{\rho}_1, \; 0 \leq - \frac{\partial r_B(U,P)}{\partial P} \leq \hat{\rho}_2, \qquad (4.39)$$

$$0 \leq - \frac{\partial K_B(T,P)}{\partial T} \leq \hat{r}_1, \; 0 \leq - \frac{\partial K_B(T,P)}{\partial P} \leq \hat{k}_2,$$

for some positive constants $\hat{\rho}$, $\hat{q}_0$, $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{k}_1$, $\hat{k}_2$, $\hat{K}_c$. Then if the following inequalities hold

$$\left[\hat{\rho} + \hat{\rho}_2 + \frac{r_{BO} K_{BO} \hat{k}_2}{\hat{K}_c^2} \right]^2 < \frac{2 r_0 r_{BO}}{3L \; K_B(\hat{T}, \hat{P})} \tag{4.40a}$$

$$\left[\frac{r_{BO} K_{BO} \hat{k}_1}{\hat{K}_c^2} + \frac{\alpha Q_m}{\delta} + \pi \nu \hat{U} \right]^2 < \frac{4 r_{BO}}{9 K_B(\hat{T}, \hat{P})} \; (\delta_0 + \alpha B^*) \tag{4.40b}$$

$$\left[\hat{\rho}_1 + \frac{\nu Q_m}{\delta} + \alpha \hat{T} \right]^2 < \frac{2 r_{BO}}{3 K_B(\hat{T}, \hat{P})} \; (\delta_1 + \nu \hat{B}) \tag{4.40c}$$

$$\left[\pi \nu + \alpha \right]^2 K_{BO}^2 < \frac{2}{3} \; (\delta_0 + \alpha \hat{B})(\delta_1 + \nu \hat{B}) \tag{4.40d}$$

$$\hat{q}_0^2 < \frac{2}{3} \frac{r_0}{L} \; (\delta_0 + \alpha \hat{B}) \tag{4.40e}$$

$\hat{E}$ is globally asymptotically stable with respect to all solutions initiating in the positive octant.

This theorem shows that in the case of population dependent emission of pollutant $(Q = Q(P))$ in the environment, the resource biomass will again settle down to its equilibrium level whose magnitude would depend upon the equilibrium level of both the population as well as the influx and washout rates of the pollutant present in the environment, the influx rate depending upon the equilibrium level of population.

CASE IV : $Q = Q(t)$, $Q(t) = Q_0 + \varepsilon \phi(t)$, $\phi(t+\omega) = \phi(t)$.

The system (2.1) can be written as

$$\dot{x} = A(x) + \varepsilon C(t), \quad x(0) = x_0 \tag{4.41}$$

where $\; \cdot = \frac{d}{dt}$, $x = [x_1, \; x_2, \; x_3, \; x_4]^{Tr} = [B, \; P, \; T, \; U]^{Tr}$,

$$A(x) = \begin{bmatrix} r_B(x_4, x_2) x_1 - r_{BO} x_1^2 / K_B(x_3, x_2) \\ r(x_1) x_2 - r_0 x_2^2 / K \\ Q_0 - \delta_0 x_3 - \alpha x_1 x_3 + \pi \nu x_1 x_4 \\ - \delta_1 x_4 + \alpha x_1 x_3 - \nu x_1 x_4 \end{bmatrix}$$

$$C(t) = (0, 0, \varepsilon\,\phi(t), 0)^{Tr}, \quad x_0 = (B(0), P(0), T(0), U(0))^{Tr},$$

Tr = transpose.

Under an analysis similar to Freedman and Shukla (1991) we can establish the following two important results of this section.

THEOREM 4.2.6 If M^* has no eigen value with zero real parts, then the system (2.1) with $Q = Q(t) = Q_0 + \varepsilon\,\phi(t)$, $\phi(t+\omega) = \phi(t)$ has a periodic solution of period ω, $(B(t,\varepsilon),\ P(t,\varepsilon),\ T(t,\varepsilon),\ U(t,\varepsilon))$ such that $(B(t,0),\ P(t,0),\ T(t,0),\ U(t,0)) = (B^*,\ P^*,\ T^*,\ U^*)$.

THEOREM 4.2.7 If M^* has no eigen value with zero real parts, then for sufficiently small ε, the stability behavior of the periodic solution of the system (4.1) is same as that of E^*.

Moreover, the periodic solution up to order ε can be computed as

$$x(t,\xi,\varepsilon) = x^* + e^{M^*t}\left[\int_0^t e^{-M^*s}C(s)\ ds \right.$$
$$\left. - (e^{M^*\omega} - 1)^{-1}\,e^{M^*\omega}\int_0^\omega e^{-M^*s}C(s)\ ds \right]\varepsilon + o(\varepsilon). \quad (4.42)$$

This shows that a periodic influx of pollutant of small amplitude causes periodic behavior in the system but the stability character remains the same as the constant case.

4.3 CONSERVATION MODEL

It is noted from the above analysis that if no effort is made to conserve the biomass and to control the population and pollution, the resource biomass may become extinct. Hence the conservation of resource is a very urgent and challenging task for scientists and engineers. To restore the sustainability of the system, some kind of efforts to conserve the resource biomass and to control population and pollution are needed. Let $F_1(t)$, $F_2(t)$,

$F_3(t)$ be the densities of efforts applied to conserve the resource biomass $B(t)$, and to control the population $P(t)$ and pollution $T(t)$ respectively. To model the system it is assumed that $F_1(t)$ is proportional to the depleted level of biomass from its carrying capacity K_{B0}, $F_2(t)$ and $F_3(t)$ are proportional to the undesired level densities of population and pollution respectively. The dynamics of the system can be governed by the following system of differential equations, Shukla et al. (1989) :

$$\frac{dB}{dt} = r_B(U,P)B - \frac{r_{B0}B^2}{K_B(T,P)} + r_{10}F_1$$

$$\frac{dP}{dt} = r(B)P - \frac{r_0P^2}{L} - r_{20}F_2P$$

$$\frac{dT}{dt} = Q - \delta_0 T - \alpha BT + \pi\nu BU - r_{30}F_3$$

$$\frac{dU}{dt} = -\delta_1 U + \alpha BT - \nu BU \qquad (4.43)$$

$$\frac{dF_1}{dt} = r_1(1 - \frac{B}{K_{B0}}) - \nu_1 F_1$$

$$\frac{dF_2}{dt} = r_2(P - P_c) - \nu_2 F_2$$

$$\frac{dF_3}{dt} = r_3(T - T_c) - \nu_3 F_3$$

$B(0) \geq 0$, $P(0) \geq 0$, $T(0) \geq 0$, $U(0) \geq kB(0)$, $F_i(0) \geq 0$, $0 \leq \pi \leq 1$, $i = 1,2,3$.

Here r_i represents the constant growth rate coefficient of the effort F_i and ν_i its depreciation rate coefficient, $i = 1,2,3$. Again r_{10} is the growth rate coefficient of the resource biomass due to the effort $F_1(t)$, r_{20} and r_{30} are the depletion rate coefficients of population and pollution due to the efforts $F_2(t)$ and $F_3(t)$ respectively. P_c is the critical value of the population which is harmless to the resource biomass and T_c is the

concentration of the pollutant permissible and harmless to the biomass. Other notations in the model (4.43) have the same meaning as in the model (4.1).

The model (4.43) is being analysed only for the case $Q = Q_0 > 0$ and $Q = Q(P)$ satisfying (4.5).

CASE I: $Q = Q_0 > 0$

In this case, it is easy to see that there exists only one equilibrium $\bar{E}(\bar{B}, \bar{P}, \bar{T}, \bar{U}, \bar{F}_1, \bar{F}_2, \bar{F}_3)$ which is given by the positive solution of the system of algebraic equations:

$$B = K_B(h^*(B), P)\left[r_B(f^*(B), P) + r_1 r_{10}(1 - B/K_{B0})/\nu_1 B\right]\Big/ r_{B0} \quad (4.44a)$$

$$P = \left[r(B) + r_2 r_{20} P_c/\nu_2\right]\Big/\left[r_0/L + r_2 r_{20}/\nu_2\right] \quad (4.44b)$$

$$T = h^*(B), \quad (4.44c)$$

$$U = f^*(B), \quad (4.44d)$$

$$F_1 = r_1(1 - B/K_{B0})/\nu_1, \quad (4.44e)$$

$$F_2 = r_2(P - P_c)/\nu_2, \quad (4.44f)$$

$$F_3 = r_3(T - T_c)/\nu_3, \quad (4.44g)$$

where

$$f^*(B) = \alpha Q_0^* B/g^*(B)$$

$$g^*(B) = \delta_0^* \delta_1 + (\delta_0^* \nu + \delta_1 \alpha)B + (1 - \pi)\nu\alpha B^2$$

$$\delta_0^* = \delta_0 + r_3 r_{30}/\nu_3$$

$$h^*(B) = [Q_0^* + \pi\nu B f^*(B)]/(\delta_0^* + \alpha B)$$

$$Q_0^* = Q_0 + r_3 r_{30} T_c/\nu_3$$

One can check that the isocline (4.44b) is an increasing function of B starting from P_s, where

$$P_s = \left[r_0 + r_2 r_{20} P_c / \nu_2 \right] \Big/ \left[r_0 / K + r_2 r_{20} / \nu_2 \right] \tag{4.45}$$

From (4.44a) one can check that this isocline is a decreasing function of B starting from B_s iff

$$r_{B0} - K_B(h^*(B),P) \left[\frac{\partial r_B}{\partial U} \frac{df^*}{dB} - \frac{r_1 r_{10}}{\nu_1 B^2} \right]$$

$$- \left[r_B(f^*(B),P) + \frac{r_1 r_{10}}{\nu_1 B} - \frac{r_1 r_{10}}{\nu_1 K_{B0}} \right] \frac{\partial K_B}{\partial T} \frac{dh^*}{dB} > 0 \tag{4.46a}$$

where B_s is a positive solution of

$$G(B) = r_{B0} B - K_B(h^*(B),0) \left[r_B(f^*(B),0) + r_1 r_{10}(1 - B/K_{B0})/\nu_1 B \right] = 0 \tag{4.46b}$$

Thus the two isoclines (4.44a,b) intersects at a unique point under condition (4.46a), their intersection value will give the B-P coordinates of $\bar{E}$ and its other coordinates can then be computed from equations (4.44c)-(4.44g).

In the following theorem, the local stability behavior of the equilibrium $\bar{E}$ is studied. The proof is similar to theorem 4.2.2 and hence is omitted.

THEOREM 4.3.1   Let the following inequalities hold:

$$\frac{r_{B0} \bar{B}}{K_B(\bar{T},\bar{P})} + \frac{r_{10} \bar{F}_1}{\bar{B}} > 2\alpha\bar{T} + r'(\bar{B})\bar{P} - (1 + \pi)\nu\bar{U} + \frac{r_1}{K_{B0}} \tag{4.47a}$$

$$\frac{r_0 \bar{P}}{L} > r_2 - \frac{r_{B0} \bar{B}^2}{K_B^2(\bar{T},\bar{P})} \frac{\partial K_B(\bar{T},\bar{P})}{\partial P} - \bar{B} \frac{\partial r_B(\bar{U},\bar{P})}{\partial P} \tag{4.47b}$$

$$\delta_0 > r_3 - \frac{r_{B0} \bar{B}^2}{K_B^2(\bar{T},\bar{P})} \frac{\partial K_B(\bar{T},\bar{P})}{\partial T} \tag{4.47c}$$

$$\delta_1 > (\pi - 1)\nu\bar{B} - \bar{B} \frac{\partial r_B(\bar{U},\bar{P})}{\partial U} \tag{4.47d}$$

$$\nu_i > r_{i0}, \quad i = 1,3. \tag{4.47e}$$

$$\nu_2 > r_{20}\bar{P} \tag{4.47f}$$

Then $\bar{E}$ is locally asymptotically stable.

The proof of the following lemma, which establishes the region of attraction for the system (4.43), is similar to lemma 4.2.1 and hence we omit it.

LEMMA 4.3.1   The set

$$\mathbb{\bar{R}} = \left\{ (B,P,T,U,F_1,F_2,F_3) \,:\, 0 \le B \le K_a, \; 0 \le P \le P_a, \; 0 \le T + U \le Q_0/\delta \right.$$

$$\left. 0 \le F_1 \le r_1/\nu_1, \; 0 \le F_2 \le r_2 P_a/\nu_2, \; 0 \le F_3 \le r_3 Q_0/\nu_3\delta \right\} \text{ is a}$$

region of attraction for all solutions initiating in the interior of positive orthant, where

$$K_a = \frac{1}{2} K_{BO}\left[1 + \left\{ 1 + 4r_1 r_{10}/\nu_1 r_{BO} K_{BO}\right\}^{1/2}\right], \quad P_a = L\, r(K_a)/r_0,$$

$$\delta = \min(\delta_0, \; \delta_1).$$

In the following theorem we have found criteria for the global stability of $\bar{E}$.

THEOREM 4.3.2   In addition to the assumptions (4.2)—(4.4), let $r(B)$, $r_B(U,P)$, $K_B(T,P)$ satisfy in $\mathbb{\bar{R}}$

$$0 \le r'(B) \le \bar{\rho}, \quad K_s \le K_B(T,P) \le K_{Bo},$$

$$0 \le -\frac{\partial r_B(U,P)}{\partial U} \le \bar{\rho}_1, \quad 0 \le -\frac{\partial r_B(U,P)}{\partial P} \le \bar{\rho}_2, \tag{4.48}$$

$$0 \le -\frac{\partial K_B(T,P)}{\partial T} \le \bar{k}_1, \quad 0 \le -\frac{\partial K_B(T,P)}{\partial P} \le \bar{k}_2,$$

for some positive constants $\bar{\rho}$, $\bar{\rho}_1$, $\bar{\rho}_2$, $\bar{k}_1$, $\bar{k}_2$, K_s. Then if the following inequalities hold

$$\left[\bar{\rho} + \bar{\rho}_2 + \frac{r_{BO} K_a \bar{k}_2}{K_s^2}\right]^2 < \frac{1}{2} \frac{r_0 r_{BO}}{L\, K_B(\bar{T},\bar{P})} \tag{4.49a}$$

$$\left[\frac{r_{BO} K_a \bar{k}_1}{K_s^2} + \frac{\alpha Q_0}{\delta} + \pi\nu\bar{U}\right]^2 < \frac{1}{3} \frac{r_{BO}}{K_B(\bar{T},\bar{P})} (\delta_0 + \alpha\bar{B}) \tag{4.49b}$$

$$\left[\bar{\rho}_1 + \frac{Q_0\nu}{\delta} + \alpha\bar{T}\right]^2 < \frac{1}{2}\frac{r_{BO}}{K_B(\bar{T},\bar{P})}(\delta_1 + \nu\bar{B}) \qquad (4.49c)$$

$$\left[\pi\nu + \alpha\right]^2 K_a^2 < \frac{2}{3}(\delta_0 + \alpha\bar{B})(\delta_1 + \nu\bar{B}) \qquad (4.49d)$$

$\bar{E}$ is globally asymptotically stable with respect to all solutions initiating in the positive orthant.

Proof: Take the following positive definite function about $\bar{E}$,

$$W(B,P,T,U,F_1,F_2,F_3) = [B - \bar{B} - \bar{B}\ln(B/\bar{B})] + [P - \bar{P} - \bar{P}\ln(P/\bar{P})]$$
$$+ \frac{1}{2}(T - \bar{T})^2 + \frac{1}{2}(U - \bar{U})^2 + \frac{1}{2}c_1(F_1 - \bar{F}_1)^2$$
$$+ \frac{1}{2}c_2(F_2 - \bar{F}_2)^2 + \frac{1}{2}c_3(F_3 - \bar{F}_3)^2 \qquad (4.50)$$

where c_1, c_2, c_3 are positive constants to be chosen suitably. Choosing $c_1 = r_{10}K_{BO}/r_1\bar{B}$, $c_2 = r_{20}/r_2$, $c_3 = r_{30}/r_3$, one can see that the time derivative of W along the solution of the system (4.43) is negative definite under the condition (4.49). Hence W is a Liapunov function with respect to $\bar{E}$, whose domain contains the region of attraction $\bar{\mathbb{R}}$, proving the theorem.

CASE II: $Q = Q(P)$

In this case, the model (4.43) has only one interior equilibrium $\bar{E}_e(\bar{B}_e,\bar{P}_e,\bar{T}_e,\bar{U}_e,\bar{F}_{1e},\bar{F}_{2e},\bar{F}_{3e})$. The coordinates of $\bar{E}_e$ can be obtained from (4.44) by replacing Q_0 by $\bar{Q}(B)$ and the existence of $\bar{E}_e$ can be seen in the similar fashion as that of $\bar{E}$ in case I of the conservation model (4.43), where $\bar{Q}(B)$ is given by

$$\bar{Q}(B) = Q(\bar{f}(B))$$

$$\bar{f}(B) = \left[r(B) + r_2 r_{20}P_c/\nu_2\right]\bigg/\left[r_0/L + r_2 r_{20}/\nu_2\right]$$

The following theorem gives the criteria for the local stability of $\bar{E}_e$ whose proof is similar to theorem 4.2.2 and hence is omitted.

THEOREM 4.3.3 Let the following inequalities hold:

$$\frac{r_{BO}\bar{B}_e}{K_B(\bar{T}_e,\bar{P}_e)} + \frac{r_{10}\bar{F}_{1e}}{\bar{B}_e} > 2\alpha\bar{T}_e + r'(\bar{B}_e)\bar{P}_e - (1 + \pi)\nu\bar{U}_e + \frac{r_1}{K_{BO}} \qquad (4.47a)$$

$$\frac{r_0\bar{P}_e}{L} > r_2 + Q'(\bar{P}_e) - \frac{r_{BO}\bar{B}_e^2}{K_B^2(\bar{T}_e,\bar{P}_e)}\frac{\partial K_B(\bar{T}_e,\bar{P}_e)}{\partial P} - \bar{B}_e\frac{\partial r_B(\bar{U}_e,\bar{P}_e)}{\partial P} \qquad (4.47b)$$

$$\delta_0 > r_3 - \frac{r_{BO}\bar{B}_e^2}{K_B^2(\bar{T}_e,\bar{P}_e)}\frac{\partial K_B(\bar{T}_e,\bar{P}_e)}{\partial T} \qquad (4.47c)$$

$$\delta_1 > (\pi - 1)\nu\bar{B}_e - \bar{B}_e\frac{\partial r_B(\bar{U}_e,\bar{P}_e)}{\partial U} \qquad (4.47d)$$

$$\nu_i > r_{10}, \quad i = 1,3. \qquad (4.47e)$$

$$\nu_2 > r_{20}\bar{P}_e \qquad (4.47f)$$

Then $\bar{E}_e$ is locally asymptotically stable.

The following lemma establishes the region of attraction for the system (4.43) whose proof is similar to lemma 4.2.1 and hence we omit it.

LEMMA 4.3.2 The set

$$\bar{R}_e = \left\{(B,P,T,U,F_1,F_2,F_3) : 0 \leq B \leq K_a, \; 0 \leq P \leq P_a, \; 0 \leq T + U \leq Q_a/\delta \right.$$

$$\left. 0 \leq F_1 \leq r_1/\nu_1, \; 0 \leq F_2 \leq r_2P_a/\nu_2, \; 0 \leq F_3 \leq r_3Q_a/\nu_3\delta\right\} \text{ is the}$$

region of attraction for all solutions initiating in the interior of positive orthant, where

$$K_a = \frac{1}{2}K_{BO}\left[1 + \left\{1 + 4r_1r_{10}/\nu_1r_{BO}K_{BO}\right\}^{1/2}\right], \quad P_a = L\,r(K_a)/r_0,$$

$$Q_a = Q(P_a), \quad \delta = \min(\delta_0, \delta_1).$$

The following theorem gives the criteria for the global stability of $\bar{E}$ whose proof is similar to theorem 4.3.2 and hence is omitted.

THEOREM 4.3.4 In addition to the assumptions (4.2)—(4.5), let $r(B)$, $Q(P)$, $r_B(U,P)$, $K_B(T,P)$ satisfy in $\bar{\mathbb{R}}_e$

$$0 \leq r'(B) \leq \bar{\rho}_e, \quad \bar{K}_e \leq K_B(T,P) \leq K_{Bo}, \quad 0 \leq Q'(P) \leq \bar{q}_e,$$

$$0 \leq -\frac{\partial r_B(U,P)}{\partial U} \leq \bar{\rho}_{1e}, \quad 0 \leq -\frac{\partial r_B(U,P)}{\partial P} \leq \bar{\rho}_{2e}, \qquad (4.48)$$

$$0 \leq -\frac{\partial K_B(T,P)}{\partial T} \leq \bar{k}_{1e}, \quad 0 \leq -\frac{\partial K_B(T,P)}{\partial P} \leq \bar{k}_{2e},$$

for some positive constants $\bar{\rho}_e$, q_e, $\bar{\rho}_{1e}$, $\bar{\rho}_{2e}$, $\bar{k}_{1e}$, $\bar{k}_{2e}$, $\bar{K}_e$. Then if the following inequalities hold

$$\left[\bar{\rho}_e + \bar{\rho}_{2e} + \frac{r_{BO}K_a\bar{k}_{2e}}{\bar{K}_e^2} \right]^2 < \frac{2}{3} \frac{r_0}{L} \frac{r_{BO}}{K_B(\bar{T}_e,\bar{P}_e)} \qquad (4.49a)$$

$$\left[\frac{r_{BO}K_a\bar{k}_{1e}}{\bar{K}_e^2} + \frac{\alpha Q_a}{\delta} + \pi\nu\bar{U}_e \right]^2 < \frac{4}{9} \frac{r_{BO}}{K_B(\bar{T}_e,\bar{P}_e)} (\delta_0 + \alpha\bar{B}_e) \qquad (4.49b)$$

$$\left[\bar{\rho}_{1e} + \frac{Q_a\nu}{\delta} + \alpha\bar{T}_e \right]^2 < \frac{2}{3} \frac{r_{BO}}{K_B(\bar{T}_e,\bar{P}_e)} (\delta_1 + \nu\bar{B}_e) \qquad (4.49c)$$

$$\left[\pi\nu + \alpha \right]^2 K_a^2 < \frac{2}{3} (\delta_0 + \alpha\bar{B}_e)(\delta_1 + \nu\bar{B}_e) \qquad (4.49d)$$

$$\bar{q}_e < \frac{2}{3} \frac{r_0}{L} (\delta_0 + \alpha\bar{B}_e)$$

$\bar{E}_e$ is globally asymptotically stable with respect to all solutions initiating in the positive orthant.

The above theorem implies that if suitable efforts are made to conserve the biomass and to control the population and pollution, an appropriate level of the forestry resource can be maintained.

4.4 EXAMPLES

In this section we give two examples to explain the applicability of the results by choosing the following functions

in model (4.1).

$$r_B(U,P) = r_{B0} - r_{B1}U - r_{B2}P$$
$$K_B(T,P) = K_{B0} - K_{B1}T - K_{B2}P \qquad (4.50)$$
$$r(B) = r_0 + \gamma_1 B$$

where the coefficients are positive.

Example 1. In this example we choose the following set of parameters in model (4.1).

$\delta_0 = 6.0$, $\delta_1 = 8.0$, $\pi = 0.5$, $\nu = 0.05$, $\alpha = 0.06$, $L = 2.0$,

$Q_0 = 30.0$, $r_{B0} = 5.5$, $r_{B1} = 0.01$, $r_{B2} = 0.02$, $K_{B0} = 15.6275$,

$K_{B1} = 0.03$, $K_{B2} = 0.04$, $r_0 = 7.0$, $\gamma_1 = 0.1$.

With the above set of parameters in model (4.1), it can be checked that the interior equilibrium $E^*(B^*,P^*,T^*,U^*)$ of the model (4.1) exists and is given by

$$B^* \simeq 15.25, \quad P^* \simeq 2.4357, \quad T^* \simeq 4.3635, \quad U^* \simeq 0.4556 .$$

It can also be checked that the conditions (4.28) in theorem 4.2.2 are satisfied and hence E^* is locally asymptotically stable.

By choosing $K_c = 2.0$ in theorem 4.2.3, it can also be checked that the conditions (4.30) in this theorem are satisfied. This shows that E^* is globally asymptotically stable.

Example 2. In this example, we choose the same functions given by (4.50) and the following set of parameters in model (4.43).

$\delta_0 = 6.0$, $\delta_1 = 8.0$, $\pi = 0.5$, $\nu = 0.05$, $\alpha = 0.06$, $L = 2.0$,

$Q_0 = 30.0$, $r_1 = 0.4$, $r_2 = 0.09$, $r_3 = 1.0$, $\nu_1 = 0.15$,

$\nu_2 = 0.2$, $\nu_3 = 0.16$, $r_{10} = 0.1$, $r_{20} = 0.03$, $r_{30} = 0.12$,

$P_c = 1.0$, $T_c = 0.5$, $r_{B0} = 6.5873$, $r_{B1} = 0.01$, $r_{B2} = 0.02$,

$K_{B0} = 15.6275$, $K_{B1} = 0.03$, $K_{B2} = 0.04$, $r_0 = 7.0$, $\gamma_1 = 0.1$.

With the above set of parameters in model (4.43), it can be checked that the interior equilibrium $\bar{E}(\bar{B},\bar{P},\bar{T},\bar{U},\bar{F}_1,\bar{F}_2,\bar{F}_3)$ of model

(4.43) exists and is given by

$$\bar{B} \simeq 15.45, \quad \bar{P} \simeq 2.4359, \quad \bar{T} \simeq 3.9778, \quad \bar{U} \simeq 0.4203,$$

$$\bar{F}_1 \simeq 0.0303, \quad \bar{F}_2 \simeq 0.6462, \quad \bar{F}_3 \simeq 21.7363 \ .$$

It can also be checked that the conditions (4.47) in theorem 4.3.1 are satisfied and hence $\bar{E}$ is locally asymptotically stable.

By choosing $K_s = 2.0$ in theorem 4.3.2, it can also be checked that the conditions (4.49) in this theorem are satisfied and hence $\bar{E}$ is globally asymptotically stable.

4.5   CONCLUSIONS

In this chapter, we have modelled the depletion of a forestry resource biomass in a forest habitat due to increase in population and pollution. It is assumed that the densities of resource biomass and population follow generalized logistic model and the growth rate of population increases as the resource biomass density increases. It is also assumed that the growth rate of the resource biomass decreases as the density of the population and the uptake concentration of pollutant increase but the corresponding carrying capacity decreases as the population density and the environmental concentration of the pollutant increase. It has been assumed further that the growth rate of uptake concentration of the pollutant by the resource biomass increases with the same amount by which the growth rate of concentration of pollutant in the environment decreases.

It has been shown that in the case of instantaneous spill of pollutant in the environment, the pollutant may wash out completely and the resource biomass will settle down to a lower equilibrium level than its carrying capacity, the magnitude being dependent on the equilibrium level of the population. It is also

noted here that even in the absence of pollutant the resource biomass may tend to extinction if the population density increases unabatedly. In the case of constant emission of pollutant it is shown that the resource biomass will settle down to a much lower equilibrium than the previous case, the magnitude of which will depend upon the equilibrium level of population and the influx and washout rates of the pollutant present in the environment. It is also noted here that if population and pollution are continued without control the resource biomass may tend to zero sooner than the case of no population present in the habitat. In the case of population dependent emission of pollutant similar results as in the constant case are found. In the case of periodic emission of pollutant, it is shown that a small periodic influx of pollutant induces a periodic behavior in the system and the stability behavior of the system is same as that of the constant emission of pollutant.

A model to conserve the resource biomass and to control population and pollution has also been proposed. It is assumed that the effort applied to conserve the biomass is proportional to the variance of biomass from its carrying capacity and the efforts applied to control the population and pollution are proportional to its respective undesired level densities. By analysing the conservation model it is shown that if suitable measures of conservation are taken the resource biomass can be maintained at an appropriate level.

The model presented in this chapter is applicable to the Doon Valley located in the northern part of Uttar Pradesh in the foothills of the Himalayas, India.
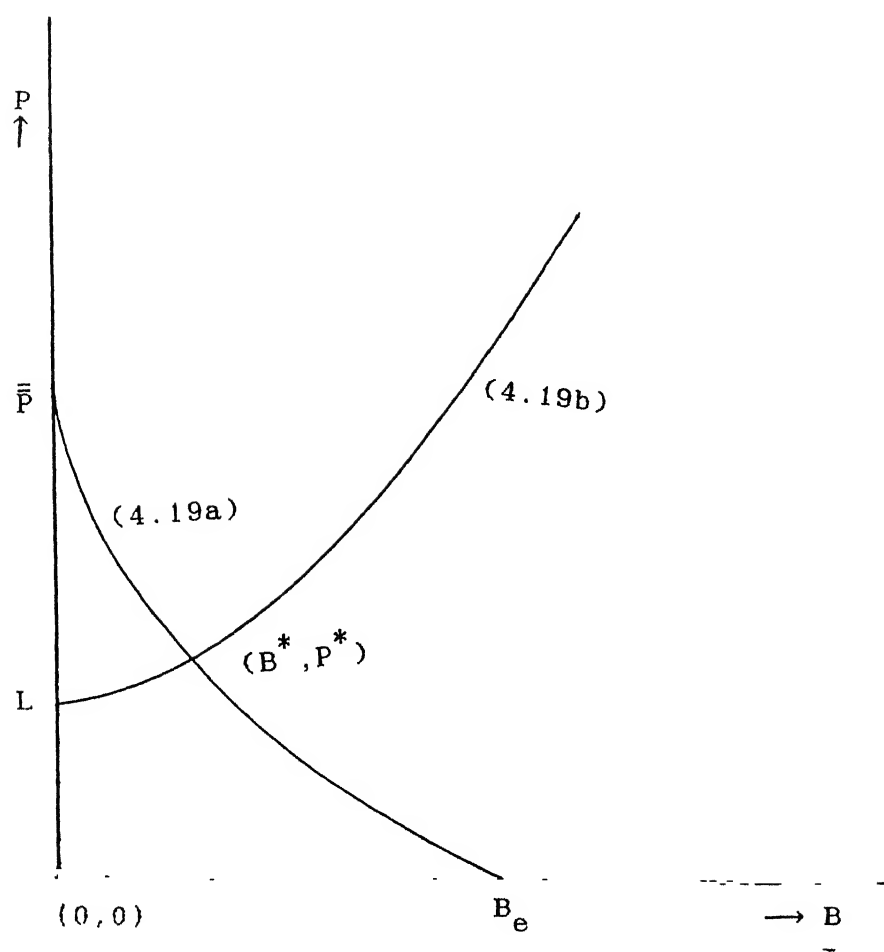
Fig. 4.1

CHAPTER V

MODELLING THE DEPLETION AND CONSERVATION OF FORESTRY RESOURCES: EFFECTS OF TWO POLLUTANTS(TOXICANTS)

5.0 INTRODUCTION

It is well known in ecological studies that ecosystems are affected by accidental spill, constant or periodic emission of toxicants in the environment, Nelson (1970), Hass (1981), Patin (1982), Jenson and Marshall (1982). Generally the effect of toxicant is to decrease the growth rate of species and the carrying capacity of the environment. In recent decades the effect of a single toxicant on various ecosystems have been studied extensively using mathematical models, Hallam and Clark (1982), Hallam et al. (1983a,b), Hallam and De Luna (1984), De Luna and Hallam (1987), Freedman and Shukla (1991). In particular, Hallam et al.(1983) studied the effect of a toxicant emitted into the environment on the population by assuming that the growth rate of population density linearly depends upon the concentration of toxicant in the population. But they did not consider the effect of environmental toxicant on the carrying capacity of the population. However, Freedman and Shukla (1991) studied the effect of a toxicant on single species and predator - prey system by considering its effect on both the growth rate of the population as well as on the habitat carrying capacity. It may be noted here that the above authors did not consider the simultaneous effect of two or more toxicants emitted into the environment on the resource.

This chapter is therefore an attempt to study the

simultaneous effects of two toxicants on a single species biological population such as forestry resource in the environment. Here the term forestry resource is used in cumulative sense for all wildlife species, forest stand biomass,etc. It is assumed that each of the toxicants is emitted into the environment with instantaneous, constant or periodic influx and is also depleted by some natural degradation factors. It is further assumed that the growth rate of uptake concentration of each of the toxicant by the species is different and is proportional to the density of the species as well as the concentrations of the toxicants present in the environment. It is considered that the growth rate of the species decreases as the uptake concentration of toxicants increases but its carrying capacity decreases with the concentrations of each of the two toxicants in the environment. Stability theory (La Salle and Lefschetz (1961)) is used to analyse the model.

We assume that all functions utilized here are sufficiently smooth that solutions to the initial value problems exist uniquely for all positive times.

5.1 MATHEMATICAL MODEL

As mentioned earlier we consider a single species population affected by two toxicants emitted into the environment in such a way that its growth rate decreases as the uptake concentration of each of the toxicants increases whereas its carrying capacity decreases due to the presence of toxicants in the environment. Using similar arguments as Freedman and Shukla (1991), the system is considered to be governed by means of the following differential equations

$$\frac{dB}{dt} = r(U_1, U_2)B - \frac{r_0 B^2}{K(T_1, T_2)}$$

$$\frac{dT_1}{dt} = Q_1(t) - \delta_1 T_1 - \alpha_1 T_1 B + \pi_1 \nu_1 B U_1$$

$$\frac{dT_2}{dt} = Q_2(t) - \delta_2 T_2 - \alpha_2 T_2 B + \pi_2 \nu_2 B U_2 \qquad (5.1)$$

$$\frac{dU_1}{dt} = -\beta_1 U_1 + \alpha_1 T_1 B - \nu_1 B U_1$$

$$\frac{dU_2}{dt} = -\beta_2 U_2 + \alpha_2 T_2 B - \nu_2 B U_2$$

$B(0) \geq 0$, $T_1(0) \geq 0$, $T_2(0) \geq 0$, $U_1(0) \geq k_1 B(0)$, $U_2(0) \geq k_2 B(0)$,

$0 \leq \pi_1 \leq 1$, $0 \leq \pi_2 \leq 1$.

Here $B(t)$ is the population density, $Q_1(t)$ and $Q_2(t)$ are the rates of introduction of toxicants with concentrations $T_1(t)$ and $T_2(t)$ respectively into the environment which are either zero or constants or periodic, $U_1(t)$ and $U_2(t)$ are the uptake concentration of the two toxicants $T_1(t)$ and $T_2(t)$ respectively by the population, $\delta_1 > 0$ and $\delta_2 > 0$ are respectively the natural washout rate coefficients of $T_1(t)$ and $T_2(t)$, α_1 and α_2 are respectively their depletion rate coefficients due to uptake by the population, β_1 and β_2 are respectively the natural washout rate coefficient of $U_1(t)$ and $U_2(t)$, ν_1 and ν_2 denote the depletion rate coefficient of $U_1(t)$ and $U_2(t)$ respectively due to dying out of some members of the populations and fractions π_1 and π_2 of this reentering into the environment, $k_1 \geq 0$ and $k_2 \geq 0$ are constants relating to the initial uptake concentration $U_1(0)$ and $U_2(0)$ with the initial population B_0.

In writing down the model (5.1) it has been assumed that the growth rate of uptake concentration $U_1(t)$ and $U_2(t)$ increases with

$\alpha_1 T_1 B$ and $\alpha_2 T_2 B$ respectively which represents the rate of depletion of respective toxicants in the environment. We assume that one toxicant is more toxic than the other, say T_2 is more toxic than T_1.

In our model (5.1), the function $r(U_1, U_2)$ represents the growth rate coefficient of biological species which decreases with U_1 and U_2. Hence we assume

$$r(0,0) = r_0 > 0, \quad \frac{\partial r}{\partial U_1} < 0, \quad \frac{\partial r}{\partial U_2} < 0 \quad \text{for } U_1 \geq 0, \, U_2 \geq 0$$

$$r(\bar{U}_1, 0) = 0, \, r(0, \bar{U}_2) = 0 \text{ for some } \bar{U}_1 > 0, \, \bar{U}_2 > 0 \tag{5.2}$$

Since T_2 is more toxic than T_1, it follows that $\bar{U}_2 < \bar{U}_1$.

Similarly the function $K(T_1, T_2)$ represents the maximum population density which the environment can support and it decreases as T_1 and T_2 increase. Hence we assume

$$K(0,0) = K_0 > 0$$

$$\frac{\partial K}{\partial T_1} < 0, \quad \frac{\partial K}{\partial T_2} < 0 \quad \text{for } T_1 \geq 0, \, T_2 \geq 0 \tag{5.3}$$

5.2 MATHEMATICAL ANALYSIS

In the following we analyse the model (5.1) for $Q_i = 0$, $Q_i = $ constant and one of the Q_i's is periodic, $i = 1, 2$.

CASE I: $Q_1(t) = Q_2(t) = 0$.

In this case the model (5.1) has two nonnegative equilibria, namely $E_0(0, 0, 0, 0)$ and $E_1(K_0, 0, 0, 0)$.

The local stability analysis of the equilibria can be studied by computing the variational matrices corresponding to each equilibrium. Let M_i be the variational matrices corresponding to E_i, $i = 0, 1$. Then we can calculate M_i's from (5.1) as

$$M_0 = \begin{bmatrix} r_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\delta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta_2 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} -r_0 & r_0 \dfrac{\partial K(0,0)}{\partial T_1} & r_0 \dfrac{\partial K(0,0)}{\partial T_2} & K_0 \dfrac{\partial r(0,0)}{\partial U_1} & K_0 \dfrac{\partial r(0,0)}{\partial U_2} \\ 0 & -\delta_1 - \alpha_1 K_0 & 0 & \pi_1 \nu_1 K_0 & 0 \\ 0 & 0 & -\delta_2 - \alpha_2 K_0 & 0 & \pi_2 \nu_2 K_0 \\ 0 & \alpha_1 K_0 & 0 & -\beta_1 - \nu_1 K_0 & 0 \\ 0 & 0 & \alpha_2 K_0 & 0 & -\beta_1 - \nu_2 K_0 \end{bmatrix}$$

From M_0, we note that E_0 is a saddle point whose unstable manifold is locally in the B direction and whose stable manifold is locally in T_1-T_2-U_1-U_2 space. From M_1, we can check up by using Routh-Hurwitz criterion (Sanchetz (1968)) that all eigen values of M_1 have negative real parts and hence E_1 is locally asymptotically stable. However, we can say much more about the stability of E_1 in the form of the following theorem.

THEOREM 5.2.1   If $B(0) > 0$, then E_1 is globally asymptotically stable.

Proof: From (5.1) we have

$$\frac{dB}{dt} = r(U_1, U_2)B - \frac{r_0 B^2}{K(T_1, T_2)}$$

$$\leq r_0 B - \frac{r_0 B^2}{K_0}$$

hence $\lim\limits_{t \to \infty} B(t) \leq K_0$

We also have

$$\frac{dT_1}{dt} + \frac{dT_2}{dt} + \frac{dU_1}{dt} + \frac{dU_2}{dt} = - \delta_1 T_1 - \delta_2 T_2 - \beta_1 U_1 - \beta_2 U_2$$

$$- (1 - \pi_1)\nu_1 BU_1 - (1 - \pi_2)\nu_2 BU_2$$

$$\leq - \delta_0 (T_1 + T_2 + U_1 + U_2)$$

where $\delta_0 = \min (\delta_1, \delta_2, \beta_1, \beta_2)$

This implies that

$$T_1(t) + T_2(t) + U_1(t) + U_2(t) \leq (T_1(0)+T_2(0)+U_1(0)+U_2(0))\ e^{-\delta_0 t}$$

and hence $\lim\limits_{t \to \infty} T_1(t) = \lim\limits_{t \to \infty} T_2(t) = \lim\limits_{t \to \infty} U_1(t) = \lim\limits_{t \to \infty} U_2(t) = 0$.

Since $T_1(t)$, $T_2(t)$, $U_1(t)$, $U_2(t)$ are all nonnegative for all $t \geq 0$. This shows that the system is dissipative and in the limit $B(t)$ is given by the solution of

$$\frac{dB}{dt} = r_0 B(1 - \frac{B}{K_0})$$

Since $B(0) > 0$, the theorem follows.

The above theorem implies that in the case of instantaneous spill of each of the two toxicants in the environment, the population with initial decrease in its density may recover back to its initial carrying capacity but time taken by this process may be very large if the washout rates of toxicants are small.

CASE II: $Q_1(t) = Q_{10} > 0$, $Q_2(t) = Q_{20} > 0$.

In this case our model (5.1) has two nonnegative equilibria, namely $E_2(0, Q_{10}/\delta_1, Q_{20}/\delta_2, 0, 0)$ and $E^*(B^*, T_1^*, T_2^*, U_1^*, U_2^*)$. We show the existence of $E^*(B^*, T_1^*, T_2^*, U_1^*, U_2^*)$ as follows:

Here B^*, T_1^*, T_2^*, U_1^* and U_2^* are the positive solution of the system of algebraic equations given below.

$$B = r(U_1, U_2) K(T_1, T_2)/r_0 \qquad (5.4a)$$

$$T_1 = \frac{Q_{10} + \pi_1 \nu_1 B U_1}{\delta_1 + \alpha_1 B} \qquad (5.4b)$$

$$T_2 = \frac{Q_{20} + \pi_2 \nu_2 B U_2}{\delta_2 + \alpha_2 B} \qquad (5.4c)$$

$$U_1 = \frac{\alpha_1 T_1 B}{\beta_1 + \nu_1 B} \qquad (5.4d)$$

$$U_2 = \frac{\alpha_2 T_2 B}{\beta_2 + \nu_2 B} \qquad (5.4e)$$

Substituting T_1 from (5.4b) in (5.4d), we get

$$U_1 = \frac{\alpha_1 Q_{10} B}{f_1(B)} = h_1(B) \quad , \text{ (say)} \qquad (5.5a)$$

where

$$f_1(B) = \beta_1 \delta_1 + (\alpha_1 \beta_1 + \nu_1 \delta_1) B + (1 - \pi_1) \alpha_1 \nu_1 B^2 > 0 \qquad (5.5b)$$

It is noted that U_1 increases as Q_{10} increases.

Substituting U_1 from (5.5a) in (5.4b), we get

$$T_1 = \frac{Q_{10} + \pi_1 \nu_1 B \, h_1(B)}{\delta_1 + \alpha_1 B} = g_1(B) \quad , \text{ (say)} \qquad (5.6)$$

which increases as Q_{10} increases.

Similarly dealing with (5.4c) and (5.4e), we get

$$U_2 = \frac{\alpha_2 Q_{20} B}{f_2(B)} = h_2(B) \quad , \text{ (say)} \qquad (5.7a)$$

$$T_2 = \frac{Q_{20} + \pi_2 \nu_2 B \, h_2(B)}{\delta_2 + \alpha_2 B} = g_2(B) \quad , \text{ (say)} \qquad (5.7b)$$

where

$$f_2(B) = \beta_2 \delta_2 + (\alpha_2 \beta_2 + \nu_2 \delta_2) B + (1 - \pi_2) \alpha_2 \nu_2 B^2 > 0 \qquad (5.7c)$$

Now to show the existence of E^*, it suffices to show that the

following two isoclines intersect at a unique point.

$$U_1 = \frac{\alpha_1 Q_{10} B}{f_1(B)} \tag{5.8a}$$

$$r(U_1, h_2(B)) = \frac{r_0 B}{K(g_1(B), g_2(B))} \tag{5.8b}$$

From (5.8a) we note the following

when $B \rightarrow 0$, $U_1 \rightarrow 0$ $\qquad$ (5.9a)

when $B \rightarrow K_0$, $U_1 \rightarrow \dfrac{\alpha_1 Q_{10} K_0}{f_1(K_0)} > 0$ $\qquad$ (5.9b)

Further,

$$\frac{dU_1}{dB} = \frac{\alpha_1 Q_{10}}{f_1^2(B)} [\beta_1 \delta_1 - (1 - \pi_1)\alpha_1 \nu_1 B^2] \tag{5.9c}$$

$$\lim_{B \to 0} \frac{dU_1}{dB} = \frac{\alpha_1 Q_{10}}{\beta_1 \delta_1} > 0 \tag{5.9d}$$

We see that $\dfrac{dU_1}{dB}$ is positive for all B iff

$$\beta_1 \delta_1 > (1 - \pi_1)\alpha_1 \nu_1 B^2 \tag{5.10}$$

Also from (5.8b) we note the following :

when $U_1 \rightarrow \bar{U}_1$, $B \rightarrow 0$ $\qquad$ (5.11a)

Also when $U_1 \rightarrow 0$, $B \rightarrow B_c$ $\qquad$ (5.11b)

where B_c is a solution of

$$r_0 B = r(0, h_2(B)) K(g_1(B), g_2(B)) \tag{5.11c}$$

Taking $F(B) = r_0 B - r(0, h_2(B)) K(g_1(B), g_2(B))$, we note that

$F(0) < 0$ and $F(K_0) > 0$ and hence there exists a B_c in the interval

$0 < B_c < K_0$ such that $F(B_c) = 0$.

We also have from (5.8b)

$$\frac{dU_1}{dB} = \frac{L_1}{L_2} = X_1 \quad , \quad (\text{say}) \tag{5.12}$$

where

$$L_1 = r_0 - K(g_1(B), g_2(B)) \frac{\partial r}{\partial U_2} \frac{dh_2}{dB} - r(U_1, h_2(B)) \left[\frac{\partial K}{\partial T_1} \frac{dg_1}{dB} + \frac{\partial K}{\partial T_2} \frac{dg_2}{dB} \right]$$

$$L_2 = K(g_1(B), g_2(B)) \frac{\partial r}{\partial U_1} < 0$$

Noticing $\lim\limits_{B \to 0} \dfrac{dg_1}{dB} = - Q_{10}\alpha_1/\delta_1^2$, $\lim\limits_{B \to 0} \dfrac{dg_2}{dB} = - Q_{20}\alpha_2/\delta_2^2$, we see that

$\dfrac{dU_1}{dB}$ is negative when $B \to 0$ and $U_1 \to \bar{U}_1$.

Further we also note from (5.12) that $\dfrac{dU_1}{dB} < 0$ for all B iff

$$L_1 > 0 \qquad\qquad\qquad (5.13)$$

Thus from the above analysis we note that the isocline (5.8a) is an increasing function of B starting from zero under the condition (5.10) and the isocline (5.8b) is a decreasing function of U_1 starting from $\bar{U}_1$ under the condition (5.13). Hence these two isoclines (5.8a) and (5.8b) must intersect at a unique point [see fig.5.1]. The intersection value of these two isoclines gives the B-U_1 coordinates of E^* and its other coordinates can be computed from (5.6) and (5.7).

Further we also have the following set of two equations:

$$U_2 = \frac{\alpha_2 Q_{20} B}{f_2(B)} \qquad\qquad\qquad (5.14a)$$

$$r(h_1(B), U_2) = \frac{r_0 B}{K(g_1(B), g_2(B))} \qquad\qquad\qquad (5.14b)$$

The interpretation of (5.14a) and (5.14b) is similar to (5.8a) and (5.8b). Here from (5.14b) we note that

$$\frac{dU_2}{dB} = \frac{M_1}{M_2} = X_2 , \quad \text{(say)} \qquad\qquad\qquad (5.15)$$

where

$$M_1 = r_0 - K(g_1(B), g_2(B)) \frac{\partial r}{\partial U_1} \frac{dh_1}{dB} - r(h_1(B), U_2)\left[\frac{\partial K}{\partial T_1} \frac{dg_1}{dB} + \frac{\partial K}{\partial T_2} \frac{dg_2}{dB}\right]$$

$$M_2 = K(g_1(B), g_2(B)) \frac{\partial r}{\partial U_2} < 0$$

We note that if

$$X_2 < X_1 < 0 \qquad\qquad (5.16)$$

then the isocline (5.14b) will be always down to the isocline (5.8b) [see fig.5.1]. It should be also noted here that the isocline (5.14b) is a decreasing function of of U_2 starting from $\bar{U}_2$ under the condition (5.16) and it intersects the B axis at $\bar{B}_c$, where $\bar{B}_c$ in the interval $0 < \bar{B}_c < K_0$ is a solution of

$$r_0 B = r(h_1(B), 0) K(g_1(B), g_2(B)) \qquad\qquad (5.17)$$

The fig.5.1 also shows that for same value of B^*, U_2^* is less than U_1^* i.e. T_2 is more toxic than T_1.

Now to study the stability behavior of E_2 and E^*, we compute the variational matrices M_2 and M^* corresponding to E_2 and E^* as follows.

$$M_2 = \begin{bmatrix} r_0 & 0 & 0 & 0 & 0 \\ -\dfrac{\alpha_1 Q_{10}}{\delta_1} & -\delta_1 & 0 & 0 & 0 \\ -\dfrac{\alpha_2 Q_{20}}{\delta_2} & 0 & -\delta_2 & 0 & 0 \\ \dfrac{\alpha_1 Q_{10}}{\delta_1} & 0 & 0 & -\beta_1 & 0 \\ \dfrac{\alpha_2 Q_{20}}{\delta_2} & 0 & 0 & 0 & -\beta_2 \end{bmatrix}$$

$$M^* = \begin{bmatrix} -\dfrac{r_0 B^*}{K(T_1^*,T_2^*)} & R_1 & R_2 & R_3 & R_4 \\[2ex] -\alpha_1 T_1^* + \pi_1 \nu_1 U_1^* & -\delta_1 - \alpha_1 B^* & 0 & \pi_1 \nu_1 B^* & 0 \\[2ex] -\alpha_2 T_2^* + \pi_2 \nu_2 U_2^* & 0 & -\delta_2 - \alpha_2 B^* & 0 & \pi_2 \nu_2 B^* \\[2ex] \alpha_1 T_1^* - \nu_1 U_1^* & \alpha_1 B^* & 0 & -\beta_1 - \nu_1 B^* & 0 \\[2ex] \alpha_2 T_2^* - \nu_2 U_2^* & 0 & \alpha_2 B_2^* & 0 & -\beta_2 - \nu_2 B_2^* \end{bmatrix}$$

where

$$R_1 = \frac{r^2(U_1^*,U_2^*)}{r_0} \frac{\partial K}{\partial T_1}(T_1^*,T_2^*), \quad R_2 = \frac{r^2(U_1^*,U_2^*)}{r_0} \frac{\partial K}{\partial T_2}(T_1^*,T_2^*),$$

$$R_3 = \frac{r^2(U_1^*,U_2^*)}{r_0} \frac{\partial r}{\partial U_1}(U_1^*,U_2^*), \quad R_4 = \frac{r^2(U_1^*,U_2^*)}{r_0} \frac{\partial r}{\partial U_2}(U_1^*,U_2^*),$$

$$(5.18)$$

From M_2, we note that E_2 is a saddle point with stable manifold locally in $T_1 - T_1 - U_1 - U_2$ space and unstable manifold locally in B direction.

We can also find sufficient conditions under which E^* is locally asymptotically stable in the form of the following theorem whose proof follows from Gershgorin's theorem (Lancaster and Tismanetsky (1985)) and hence omitted.

THEOREM 5.2.2 Let the following inequalities hold

$$\frac{r_0 B^*}{K(T_1^*,T_2^*)} > 2(\alpha_1 T_1^* + \alpha_2 T_2^*) - (1 + \pi_1)\nu_1 U_1^* - (1 + \pi_2)\nu_2 U_2^* \qquad (5.19a)$$

$$r_0 \delta_1 > - r^2(U_1^*,U_2^*) \frac{\partial K}{\partial T_1}(T_1^*,T_2^*) \qquad (5.19b)$$

$$r_0 \delta_2 > - r^2(U_1^*,U_2^*) \frac{\partial K}{\partial T_2}(T_1^*,T_2^*) \qquad (5.19c)$$

$$\beta_1 + (1 - \pi_1)\nu_1 B^* > - B^* \frac{\partial r}{\partial U_1}(U_1^*,U_2^*) \qquad (5.19d)$$

$$\beta_2 + (1 - \pi_2)\nu_2 B^* > - B^* \frac{\partial r}{\partial U_2} (U_1^*, U_2^\dagger) \qquad (5.19e)$$

Then E^* is locally asymptotically stable.

Now to show that E^* is globally asymptotically stable, we first need a lemma which establishes the region of attraction for the system (5.1).

LEMMA 5.2.1   The set

$$\mathbb{R} = \left\{ (B, T_1, T_2, U_1, U_2): \ 0 \le B \le K_0, \ 0 \le T_1 + T_2 + U_1 + U_2 \le Q_0/\delta_0, \right.$$

where $\delta_0 = \min(\delta_1, \delta_2, \beta_1, \beta_2)$, $Q_0 = Q_{10} + Q_{20}$ is a region of attraction for all solutions initiating in the positive octant.

Proof: As before, $\lim\limits_{t \to \infty} B(t) \le K_0$

and $\dfrac{dT_1}{dt} + \dfrac{dT_2}{dt} + \dfrac{dU_1}{dt} + \dfrac{dU_2}{dt} \le - \delta_0 (T_1 + T_2 + U_1 + U_2) + Q_0$

Hence $\lim\limits_{t \to \infty} [T_1(t) + T_2(t) + U_1(t) + U_2(t)] \le Q/\delta_0$, proving the lemma.

THEOREM 5.2.3   In addition to the assumptions (5.2) and (5.3), let $r(U_1, U_2)$ and $K(T_1, T_2)$ satisfy in $\mathbb{R}$

$$K_m \le K(T_1, T_2) \le K_0, \quad 0 \le - \frac{\partial r}{\partial U_1} (U_1, U_2) \le \rho_1, \quad 0 \le - \frac{\partial r}{\partial U_2} (U_1, U_2) \le \rho_2$$

$$0 \le - \frac{\partial K}{\partial T_1} (T_1, T_2) \le k_1, \quad 0 \le - \frac{\partial K}{\partial T_2} (T_1, T_2) \le k_2 \qquad (5.20)$$

for some positive constants K_m, ρ_1, ρ_2, k_1, k_2.

Then if the following inequalities hold

$$\left[\frac{r_0 k_1 K_0}{K_m^2} + \frac{\alpha_1 Q_0}{\delta_0} + \pi_1 \nu_1 U_1^* \right]^2 < \frac{1}{2} \frac{r_0}{K(T_1^*, T_2^*)} (\delta_1 + \alpha_1 B^*) \qquad (5.21a)$$

$$\left[\frac{r_0 k_2 K_0}{K_m^2} + \frac{\alpha_2 Q_0}{\delta_0} + \pi_2 \nu_2 U_2^* \right]^2 < \frac{1}{2} \frac{r_0}{K(T_1^*, T_2^*)} (\delta_2 + \alpha_2 B^*) \qquad (5.21b)$$

$$\left[\rho_1 + \frac{\nu_1 Q_0}{\delta_0} + \alpha_1 T_1^* \right]^2 < \frac{1}{2} \frac{r_0}{K(T_1^*, T_2^*)} (\beta_1 + \nu_1 B^*) \qquad (5.21c)$$

$$\left[\rho_2 + \frac{\nu_2 Q_0}{\delta_0} + \alpha_2 T_2^* \right]^2 < \frac{1}{2} \frac{r_0}{K(T_1^*, T_2^*)} (\beta_2 + \nu_2 B^*) \qquad (5.21d)$$

$$\left[\pi_1 \nu_1 + \alpha_1 \right]^2 K_0^2 < (\delta_1 + \alpha_1 B^*)(\beta_1 + \nu_1 B^*) \qquad (5.21e)$$

$$\left[\pi_2 \nu_2 + \alpha_2 \right]^2 K_0^2 < (\delta_2 + \alpha_2 B^*)(\beta_2 + \nu_2 B^*) \qquad (5.21f)$$

E^* is globally asymptotically stable with respect to all solutions initiating in the positive octant.

Proof: We consider the following positive definite function about E^* ,

$$V(B, T_1, T_2, U_1, U_2) = (B - B^* - B^* \ln \frac{B}{B^*}) + \frac{1}{2} (T_1 - T_1^*)^2$$

$$+ \frac{1}{2} (T_2 - T_2^*)^2 + \frac{1}{2} (U_1 - U_1^*)^2 + \frac{1}{2} (U_2 - U_2^*)^2 \qquad (5.22)$$

Differentiating V with respect to t along the solution of the model (5.1), we get

$$\frac{dV}{dt} = (B - B^*) \left[r(U_1, U_2) - \frac{r_0 B}{K(T_1, T_2)} \right]$$

$$+ (T_1 - T_1^*) \left[Q_{10} - \delta_1 T_1 - \alpha_1 T_1 B + \pi_1 \nu_1 B U_1 \right]$$

$$+ (T_2 - T_2^*) \left[Q_{20} - \delta_2 T_2 - \alpha_2 T_2 B + \pi_2 \nu_2 B U_2 \right]$$

$$+ (U_1 - U_1^*) \left[- \beta_1 U_1 + \alpha_1 T_1 B - \nu_1 B U_1 \right]$$

$$+ (U_2 - U_2^*) \left[- \beta_2 U_2 + \alpha_2 T_2 B - \nu_2 B U_2 \right] \qquad (5.23)$$

Using (5.4) and simplifying, we get

$$\frac{dV}{dt} = \frac{r_0}{K(T_1^*, T_2^*)} (B - B^*)^2 - (\delta_1 + \alpha_1 B^*)(T_1 - T_1^*)^2$$

$$- (\delta_2 + \alpha_2 B^*)(T_2 - T_2^*)^2 - (\beta_1 + \nu_1 B^*)(U_1 - U_1^*)^2$$

$$- (\beta_2 + \nu_2 B^*)(U_2 - U_2^*)^2$$

$$+ (B - B^*)(T_1 - T_1^*)[- r_0 B \xi_1(T_1, T_2) - \alpha_1 T_1 + \pi_1 \nu_1 U_1^*]$$

$$+ (B - B^*)(T_2 - T_2^*)[- r_0 B \xi_2(T_1^*, T_2) - \alpha_2 T_2 + \pi_2 \nu_2 U_2^*]$$

$$+ (B - B^*)(U_1 - U_1^*)[\eta_1(U_1, U_2) - \nu_1 U_1 + \alpha_1 T_1^*]$$

$$+ (B - B^*)(U_2 - U_2^*)[\eta_2(U_1^*, U_2) - \nu_2 U_2 + \alpha_2 T_2^*]$$

$$+ (T_1 - T_1^*)(U_1 - U_1^*)[\pi_1 \nu_1 B + \alpha_1 B]$$

$$+ (T_2 - T_2^*)(U_2 - U_2^*)[\pi_2 \nu_2 B + \alpha_2 B] \qquad (5.24)$$

where

$$\eta(U_1, U_2) = \begin{cases} [r(U_1, U_2) - r(U_1^*, U_2)]/(U_1 - U_1^*), & U_1 \neq U_1^* \\[2mm] \dfrac{\partial r}{\partial U_1}(U_1, U_2), & U_1 = U_1^* \end{cases} \qquad (5.25a)$$

$$\eta(U_1^*, U_2) = \begin{cases} [r(U_1^*, U_2) - r(U_1^*, U_2^*)]/(U_2 - U_2^*), & U_2 \neq U_2^* \\[2mm] \dfrac{\partial r}{\partial U_2}(U_1^*, U_2), & U_2 = U_2^* \end{cases} \qquad (5.25b)$$

$$\xi_1(T_1, T_2) = \begin{cases} [\dfrac{1}{K(T_1, T_2)} - \dfrac{1}{K(T_1^*, T_2)}]/(T_1 - T_1^*), & T_1 \neq T_1^* \\[3mm] - \dfrac{1}{K^2(T_1^*, T_2)} \dfrac{\partial K}{\partial T_2}(T_1^*, T_2), & T_1 = T_1^* \end{cases} \qquad (5.25c)$$

$$\xi_2(T_1',T_2) = \begin{cases} [\dfrac{1}{K(T_1^*,T_2)} - \dfrac{1}{K(T_1^*,T_2^*)}]/(T_2 - T_2^*), & T_2 \neq T_2^* \\[3mm] -\dfrac{1}{K^2(T_1^*,T_2^*)}\dfrac{\partial K}{\partial T_2}(T_1^*,T_2^*), & T_2 = T_2^* \end{cases} \qquad (5.25d)$$

From (5.20) and the mean value theorem, we note that

$$|\eta(U_1,U_2)| \leq \rho_1, \quad |\eta(U_1^*,U_2)| \leq \rho_2, \quad |\xi_1(T_1,T_2)| \leq k_1/K_m^2 \quad \text{and}$$

$$|\xi_2(T_1^*,T_2)| \leq k_2/K_m^2 \qquad (5.26)$$

$\dfrac{dV}{dt}$ can further be written as sum of the quadratics

$$\frac{dV}{dt} = -\frac{1}{2}a_{11}(B-B^*)^2 + a_{12}(B-B^*)(T_1-T_1^*) - \frac{1}{2}a_{22}(T_1-T_1^*)^2$$

$$-\frac{1}{2}a_{11}(B-B^*)^2 + a_{13}(B-B^*)(T_2-T_2^*) - \frac{1}{2}a_{33}(T_2-T_2^*)^2$$

$$-\frac{1}{2}a_{11}(B-B^*)^2 + a_{14}(B-B^*)(U_1-U_1^*) - \frac{1}{2}a_{44}(U_1-U_1^*)^2$$

$$-\frac{1}{2}a_{11}(B-B^*)^2 + a_{15}(B-B^*)(U_2-U_2^*) - \frac{1}{2}a_{55}(U_2-U_2^*)^2$$

$$-\frac{1}{2}a_{22}(T_1-T_1^*)^2 + a_{24}(T_1-T_1^*)(U_1-U_1^*) - \frac{1}{2}a_{44}(U_1-U_1^*)^2$$

$$-\frac{1}{2}a_{33}(T_2-T_2^*)^2 + a_{35}(T_2-T_2^*)(U_2-U_2^*) - \frac{1}{2}a_{55}(U_2-U_2^*)^2$$

$$(5.27)$$

where

$$a_{11} = \frac{1}{2}\frac{r_o}{K(T_1^*,T_2^*)}, \quad a_{22} = \delta_1 + \alpha_1 B^*, \quad a_{33} = \delta_2 + \alpha_2 B^*,$$

$$a_{44} = \beta_1 + \nu_1 B^*, \quad a_{55} = \beta_2 + \nu_2 B^*,$$

$$a_{12} = -r_0 B\xi_1(T_1,T_2) - \alpha_1 T_1 + \pi_1\nu_1 U_1^*,$$

$$a_{13} = -r_0 B\xi_2(T_1^*,T_2) - \alpha_2 T_2 + \pi_2\nu_2 U_2^*,$$

$$a_{14} = \eta_1(U_1,U_2) - \nu_1 U_1 + \alpha_1 T_1^*,$$

$$a_{15} = \eta_1(U_1^*,U_2) - \nu_2 U_2 + \alpha_2 T_2^*,$$

$$a_{24} = (\pi_1 \nu_1 + \alpha_1)B, \quad a_{35} = (\pi_2 \nu_2 + \alpha_2)B .$$

The sufficient conditions for $\frac{dV}{dt}$ to be negative definite are that the following inequalities hold:

$$a_{12}^2 < a_{11}a_{22} \tag{5.28a}$$

$$a_{13}^2 < a_{11}a_{33} \tag{5.28b}$$

$$a_{14}^2 < a_{11}a_{44} \tag{5.28c}$$

$$a_{15}^2 < a_{11}a_{55} \tag{5.28d}$$

$$a_{24}^2 < a_{22}a_{44} \tag{5.28e}$$

$$a_{35}^2 < a_{33}a_{55} \tag{5.28f}$$

We note that (5.21a,b,c,d,e,f) ⇒ (5.28a,b,c,d,e,f) respectively. Hence V is a Liapunov function with respect to E^* whose domain contains the region of attraction $\mathbb{R}$, proving the theorem.

The above theorem shows that when the inequalities (5.21) hold, the population will settle down to a lower equilibrium level than its initial carrying capacity, the magnitude of which will depend upon the emission and washout rates of the two toxicants and will be less than the case of a single toxicant, other parameters being the same in the system.

REMARK: When either $Q_1(t) = 0$ or $Q_2(t) = 0$, then the corresponding model can be analysed as in case II and the stability results are found to be similar to this. In particular, the equilibrium level of the population will be higher than the case II for the same values of parameters and the corresponding non zero equilibrium is stable under suitably modified conditions.

CASE III   Let one of the Q_i is periodic and other is constant, say $Q_1(t) = Q_{10} + \varepsilon\phi(t)$, $\phi(t+\omega) = \phi(t)$, $Q_2(t) = Q_{20}$.

The system (5.1) can be written as

$$\frac{dx}{dt} = A(x) + \varepsilon C(t), \quad x(0) = x_0. \tag{5.29}$$

where

$$x = (x_1, x_2, x_3, x_4, x_5)^{Tr} = (B, T_1, T_2, U_1, U_2)^{Tr},$$

Tr = Transpose

$$A(x) = \begin{bmatrix} r(x_4,x_5)x_1 - r_0 x_1^2/K(x_2,x_3) \\ Q_{10} - \delta_1 x_2 - \alpha_1 x_2 x_1 + \pi_1 \nu_1 x_1 x_4 \\ Q_{20} - \delta_2 x_3 - \alpha_2 x_3 x_1 + \pi_2 \nu_2 x_1 x_5 \\ -\beta_1 x_4 + \alpha_1 x_2 x_1 - \nu_1 x_1 x_4 \\ -\beta_2 x_5 + \alpha_2 x_3 x_1 - \nu_2 x_1 x_5 \end{bmatrix}$$

$$C(t) = (0, \varepsilon\phi(t), 0, 0, 0)^{Tr}, \quad x_0 = (B_0, T_{10}, T_{20}, U_{10}, U_{20})^{Tr},$$

Let $x(t,\xi,\varepsilon)$ be the solution of (5.29) such that $x(0,\xi,\varepsilon) = x^* + \xi$,

where

$$x = (B^*, T_1^*, T_2^*, U_1^*, U_2^*)^{Tr}, \quad \xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)^{Tr}$$

Using the similar analysis as in Freedman and Shukla (1991), we can state the following theorems.

THEOREM 5.2.4   If M^* has no eigen value with zero real parts, then the system (5.1) with $Q_1(t) = Q_{10} + \varepsilon\phi(t)$, $Q_2(t) = Q_{20}$ has a periodic solution of period ω, $(B(t,\varepsilon), T_1(t,\varepsilon), T_2(t,\varepsilon), U_1(t,\varepsilon), U_2(t,\varepsilon))$ such that $(B(t,0), T_1(t,0), T_2(t,0), U_1(t,0), U_2(t,0)) = (B^*, T_1^*, T_2^*, U_1^*, U_2^*)$.

THEOREM 5.2.5   If M^* has no eigen value with zero real parts, then for sufficiently small ε, the stability behavior of the periodic solution of the system (5.1) is same as that of E^*.

Further the periodic solution up to order ε can be computed

as

$$x(t,\xi,\varepsilon) = z^* + e^{M^*t} \left[\int_0^t e^{-M^*s} C(s)\ ds \right.$$

$$\left. - (e^{M^*\omega} - I)^{-1} e^{M^*\omega} \int_0^\omega e^{-M^*s} C(s)\ ds \right] \varepsilon + o(\varepsilon) \qquad (5.30)$$

REMARK: If $Q_1(t)$ and $Q_1(t)$ both are periodic of the same period, then the model can be analysed in the similar fashion.

5.3 PARTICULAR CASE

Let T and U are the cumulative concentrations of toxicants present in the environment and in the population respectively. Then taking $\pi_i = \pi$, $\alpha_i = \alpha$, $\beta_i = \beta$, $\delta_i = \delta$, $\nu_i = \nu$, $i = 1$, 2; then system (5.1) reduces to Freedman and Shukla(1991) :

$$\frac{dB}{dt} = r(U)B - \frac{r_0 B^2}{K(T)}$$

$$\frac{dT}{dt} = Q(t) - \delta T - \alpha TB + \pi\nu BU \qquad (5.31)$$

$$\frac{dU}{dt} = -\beta U + \alpha TB - \nu BU$$

$B(0) \geq 0$, $T(0) \geq 0$, $U(0) \geq cB(0)$, $0 \leq \pi \leq 1$.

In the case $Q(t) = Q_0$, it can be seen that the interior equilibrium $E^*(B^*, T^*, U^*)$ exists if and only if

$$r(U) \frac{\partial K(g(B))}{\partial g} \frac{dg}{dB} < r_0 \qquad (5.32a)$$

$$\beta\delta > (1 - \pi)\nu\alpha B^2 \qquad (5.32b)$$

where

$$T = g(B) = \frac{Q_0 + \pi\nu B\ h(B)}{\delta + \alpha B} \qquad (5.33a)$$

$$U = h(B) = \alpha Q_0 B \left/ \left[\beta\delta + (\alpha\beta + \nu\delta)B + (1 - \pi)\nu\alpha B^2 \right] \right. \qquad (5.33b)$$

It should be noted here that the conditions (5.32a) and (5.32b)

for the existence of interior equilibrium are more general than the case of Freedman and Shukla(1991). Other results corresponding to this case are known in Freedman and Shukla(1991).

5.4   CONSERVATION MODEL

In the previous sections, it is noted that as the concentration as well as the number of toxicants increase the density of population decreases. It is therefore essential to conserve the species and to control the toxicants. Let F be the density of effort applied to conserve the species which is proportional to the depleted level of species from its carrying capacity K_0. Let F_1 and F_2 be the densities of efforts applied to control the toxicants $T_1(t)$ and $T_2(t)$ respectively which are proportional to the undesired level of toxicants. Then the system can be governed by the following system of differential equations

$$\frac{dB}{dt} = r(U_1,U_2)B - \frac{r_0 B^2}{K(T_1,T_2)} + r_{11}F$$

$$\frac{dT_1}{dt} = Q_1(t) - \delta_1 T_1 - \alpha_1 T_1 B + \pi_1 \nu_1 BU_1 - r_{10}F_1$$

$$\frac{dT_2}{dt} = Q_2(t) - \delta_2 T_2 - \alpha_2 T_2 B + \pi_2 \nu_2 BU_2 - r_{20}F_2$$

$$\frac{dU_1}{dt} = - \beta_1 U_1 + \alpha_1 T_1 B - \nu_1 BU_1 \qquad\qquad (5.34)$$

$$\frac{dU_2}{dt} = - \beta_2 U_2 + \alpha_2 T_2 B - \nu_2 BU_2$$

$$\frac{dF}{dt} = \gamma(1 - \frac{B}{K_0}) - \gamma_0 F$$

$$\frac{dF_1}{dt} = \gamma_1(T_1 - T_{c1}) - \gamma_{10}F_1$$

$$\frac{dF_2}{dt} = \gamma_2(T_2 - T_{c2}) - \gamma_{20}F_2$$

$B(0) \geq 0$, $T_1(0) \geq 0$, $T_2(0) \geq 0$, $U_1(0) \geq k_1 B(0)$, $U_2(0) \geq k_2 B(0)$, $F(0) \geq 0$, $F_1(0) \geq 0$, $F_2(0) \geq 0$, $0 \leq \pi_1 \leq 1$, $0 \leq \pi_2 \leq 1$.

Here γ, γ_1, γ_2 are the growth rate coefficients of the efforts F, F_1, F_2 respectively and γ_0, γ_{10}, γ_{20} are their respective depletion rate coefficients. r_{11} is the growth rate coefficient of $B(t)$ due to the effort $F(t)$ and r_{10}, r_{20} are respectively the depletion rate coefficient of $T_1(t)$, $T_2(t)$ due to the effort $F_1(t)$, $F_2(t)$. Other notations have the same meaning as the model (5.1). The model (5.34) is analysed here only for the case when $Q_1(t) = Q_{j0} > 0$.

It can be checked that there is only one interior equilibrium $\hat{E}(\hat{B}, \hat{T}_1, \hat{T}_2, \hat{U}_1, \hat{U}_2, \hat{F}, \hat{F}_1, \hat{F}_2)$. The coordinates of $\hat{E}$ are the positive solution of the system of algebraic equations given below.

$$r_0 B = K(\hat{g}_1(B), \hat{g}_2(B))\left[r(U_1, \hat{h}_2(B)) + \frac{r_{11}\gamma}{\gamma_0 B}(1 - \frac{B}{K_0})\right] \qquad (5.35a)$$

$$T_1 = \frac{\hat{Q}_{10}\hat{f}_1(B) + \pi_1 \nu_1 \alpha_1 Q_{10} B^2}{\hat{f}_1(B)(\delta_1 + \alpha_1 B)} = \hat{g}_1(B) \qquad \text{(say)} \qquad (5.35b)$$

$$T_2 = \frac{\hat{Q}_{20}\hat{f}_2(B) + \pi_2 \nu_2 \alpha_2 Q_{20} B^2}{\hat{f}_2(B)(\delta_2 + \alpha_2 B)} = \hat{g}_2(B) \qquad \text{(say)} \qquad (5.35c)$$

$$U_1 = \frac{\alpha_1 Q_{10} B}{\hat{f}_1(B)} = \hat{h}_1(B) \qquad \text{(say)} \qquad (5.35d)$$

$$U_2 = \frac{\alpha_2 Q_{20} B}{\hat{f}_2(B)} = \hat{h}_2(B) \qquad \text{(say)} \qquad (5.35e)$$

$$F = \gamma(1 - B/K_0)/\gamma_0, \quad F_1 = \gamma_1(T_1 - T_{c1})/\gamma_{10}, \quad F_2 = \gamma_2(T_2 - T_{c2})/\gamma_{20}$$
$$(5.35f)$$

where

$$\hat{f}_1(B) = \beta_1 \hat{\delta}_1 + (\alpha_1 \beta_1 + \nu_1 \hat{\delta}_1)B + (1 - \pi_i)\alpha_1 \nu_i B^2 > 0$$

$$\hat{Q}_{i0} = Q_{10} + \gamma_i r_{10}/\gamma_{10}, \quad \hat{\delta}_1 = \delta_1 + \gamma_i r_{10}/\gamma_{i0}, \quad i = 1, 2.$$

It can be checked that the isocline (5.35a) and (5.35d) intersect at a unique point in $B - U_1$ plane, provided

$$\beta_1 \delta_1 > (1 - \pi_1)\alpha_1 \nu_1 B^2 \tag{5.36a}$$

$$r_0 - K(\hat{g}_1(B), \hat{g}_2(B)) \left[\frac{\partial r}{\partial h_2} \frac{dh_2}{dB} - \frac{r_{11} \gamma}{\gamma_0 B^2} \right]$$

$$- \left[r(U_1, \hat{h}_2(B)) + \frac{r_{11}\gamma}{\gamma_0 B}(1 - \frac{B}{K_0}) \right] \left[\frac{\partial K}{\partial T_1} \frac{d\hat{g}_1}{dB} + \frac{\partial K}{\partial T_2} \frac{d\hat{g}_2}{dB} \right] > 0 \tag{5.36b}$$

The following theorem gives the criteria for the local stability of $\hat{E}$ whose proof is similar to theorem 5.2.2 and hence is omitted.

THEOREM 5.4.1 Let the following inequalities hold

$$\frac{r_0 \hat{B}}{K(\hat{T}_1, \hat{T}_2)} + \frac{r_{11} \hat{F}}{\hat{B}} > 2(\alpha_1 \hat{T}_1 + \alpha_2 \hat{T}_2) - (1 + \pi_1)\nu_1 \hat{U}_1$$

$$- (1 + \pi_1)\nu_1 \hat{U}_1 + \frac{\gamma}{K_0} \tag{5.37a}$$

$$\delta_1 > \gamma_1 - \frac{r_0 \hat{B}}{K^2(\hat{T}_1, \hat{T}_2)} \frac{\partial K}{\partial T_1}(\hat{T}_1, \hat{T}_2) \tag{5.37b}$$

$$\delta_2 > \gamma_2 - \frac{r_0 \hat{B}}{K^2(\hat{T}_1, \hat{T}_2)} \frac{\partial K}{\partial T_2}(\hat{T}_1, \hat{T}_2) \tag{5.37c}$$

$$\beta_1 + (1 - \pi_1)\nu_1 \hat{B} > - \hat{B} \frac{\partial r}{\partial U_1}(\hat{U}_1, \hat{U}_2) \tag{5.37d}$$

$$\beta_2 + (1 - \pi_2)\nu_2 \hat{B} > - \hat{B} \frac{\partial r}{\partial U_2}(\hat{U}_1, \hat{U}_2) \tag{5.37e}$$

$$\gamma_0 > r_{11}, \quad \gamma_{10} > r_{10}, \quad \gamma_{20} > r_{20} \tag{5.37f}$$

Then $\hat{E}$ is locally asymptotically stable.

The following lemma establishes the region of attraction for the system (5.34) whose proof is similar to lemma 5.2.1.

LEMMA 5.4.1 The set

$$\hat{\mathbb{R}} = \left\{ (B, T_1, T_2, U_1, U_2, F, F_1, F_2) : 0 \leq B \leq K_a, \ 0 \leq \sum_{i=1}^{2} (T_1 + U_1) \leq \frac{Q_0}{\delta_0}, \right.$$

$$\left. 0 \leq F \leq \frac{\gamma}{\gamma_0}, \ 0 \leq F_1 \leq \frac{\gamma_1 Q_0}{\delta_0 \gamma_{10}}, \ 0 \leq F_2 \leq \frac{\gamma_2 Q_0}{\delta_0 \gamma_{20}} \right\} \text{ is a region of}$$

attraction for all solutions initiating in the positive octant, where

$$\delta_0 = \min (\delta_1, \ \delta_2, \ \beta_1, \ \beta_2), \ Q_0 = Q_{10} + Q_{20}.$$

$$K_a = \frac{K_0}{2} \left[1 + \left\{ 1 + \frac{4 r_{11} \gamma}{\gamma_0 r_0 K_0} \right\}^{1/2} \right].$$

In the following theorem, the conditions for $\hat{E}$ to be globally asymptotically stable has been found.

THEOREM 5.4.2 In addition to the assumptions (5.2) and (5.3), let $r(U_1, U_2)$ and $K(T_1, T_2)$ satisfy in $\hat{\mathbb{R}}$

$$\hat{K}_m \leq K(T_1, T_2) \leq K_0, \ 0 \leq -\frac{\partial r}{\partial U_i} \leq \hat{\rho}_i, \ 0 \leq -\frac{\partial K}{\partial T_i} \leq \hat{k}_i \qquad (5.38)$$

for all $T_i \geq 0$, $U_i \geq 0$ and for some positive constants $\hat{K}_m$, $\hat{\rho}_i$, $\hat{k}_i$, $i = 1, 2$.

Then if the following inequalities hold

$$\left[\frac{r_0 \hat{k}_1 K_a}{\hat{K}_m^2} + \frac{\alpha_1 Q_0}{\delta_0} + \pi_1 \nu_1 \hat{U}_1 \right]^2 < \frac{1}{2} \frac{r_0}{K(\hat{T}_1, \hat{T}_2)} (\delta_1 + \alpha_1 \hat{B}) \qquad (5.39a)$$

$$\left[\frac{r_0 \hat{k}_2 K_a}{\hat{K}_m^2} + \frac{\alpha_2 Q_0}{\delta_0} + \pi_2 \nu_2 \hat{U}_2 \right]^2 < \frac{1}{2} \frac{r_0}{K(\hat{T}_1, \hat{T}_2)} (\delta_2 + \alpha_2 \hat{B}) \qquad (5.39b)$$

$$\left[\hat{\rho}_1 + \frac{\nu_1 Q_0}{\delta_0} + \alpha_1 \hat{T}_1 \right]^2 < \frac{1}{2} \frac{r_0}{K(\hat{T}_1, \hat{T}_2)} (\beta_1 + \nu_1 \hat{B}) \qquad (5.39c)$$

$$\left[\hat{\rho}_2 + \frac{\nu_2 Q_0}{\delta_0} + \alpha_2 \hat{T}_2\right]^2 < \frac{1}{2} \frac{r_0}{K(\hat{T}_1, \hat{T}_2)} (\beta_2 + \nu_2 \hat{B}) \tag{5.39d}$$

$$\left[\pi_1 \nu_1 + \alpha_1\right]^2 K_a^2 < (\delta_1 + \alpha_1 \hat{B})(\beta_1 + \nu_1 \hat{B}) \tag{5.39e}$$

$$\left[\pi_2 \nu_2 + \alpha_2\right]^2 K_a^2 < (\delta_2 + \alpha_2 \hat{B})(\beta_2 + \nu_2 \hat{B}) \tag{5.39f}$$

$\hat{E}$ is globally asymptotically stable with respect to all solutions initiating in the positive octant.

Proof: Taking the following positive definite function about $\hat{E}$,

$$W = (B - \hat{B} - \hat{B} \ln \frac{B}{\hat{B}}) + \frac{1}{2} \sum_{i=1}^{2} (T_i - \hat{T}_i)^2 + \frac{1}{2} \sum_{i=1}^{2} (U_i - \hat{U}_i)^2$$

$$+ \frac{1}{2} \frac{r_{11} K_0}{\gamma \hat{B}} (F - \hat{F})^2 + \frac{1}{2} \sum_{i=1}^{2} \frac{r_{10}}{\gamma_i} (F_i - \hat{F}_i)^2 \tag{5.40}$$

one can see that the derivative of W with respect to t along the solution of (5.34) under the condition (5.39) is negative definite, and thus the theorem follows.

The above theorem shows that if suitable measures to conserve the species and to control the toxicants are taken, an appropriate level of the species population can be maintained.

5.5   EXAMPLES

To explain the applicability of the results discussed above we consider the following examples for the case of two toxicants affecting the single species population by choosing

$$r(U_1, U_2) = r_0 - \frac{a_1 U_1}{1 + r_1 U_1} - \frac{a_2 U_2}{1 + r_2 U_2} \tag{5.41a}$$

$$K(T_1, T_2) = K_0 - \frac{b_1 T_1}{1 + m_1 T_1} - \frac{b_2 T_2}{1 + m_2 T_2} \tag{5.41b}$$

where the coefficients are positive.

Example 1.

In the above two equations (5.41a) and (5.41b), choose

$a_1 = a_2 = b_1 = b_2 = 1$, $r_1 = 0.2$, $r_2 = 0.1$, $m_1 = 0.02$,

$m_2 = 0.01$, $r_0 = 4$, $K_0 = 12.3859$.

Now we choose the following set of parameters:

$\alpha_1 = 0.02$, $\alpha_2 = 0.01$, $\beta_1 = 10$, $\beta_2 = 12$, $\delta_1 = 14$, $\delta_2 = 15$,

$\nu_1 = 0.03$, $\nu_2 = 0.04$, $\pi_1 = 0.05$, $\pi_2 = 0.06$, $Q_{10} = 50$. $Q_{20} = 40$.

By choosing the above set of parameters in the model (5.1), it can be checked that all the conditions for the existence of E^* are satisfied and thus E^* can be found as follows.

$B^* \simeq 6.4$, $T_1^* \simeq 3.5391$, $T_2^* \simeq 2.6554$, $U_1^* = 0.0444$, $U_2^* \simeq 0.0139$.

It can also be checked that all the conditions (5.19) in theorem 5.2.2 are satisfied for the above set of parameters and hence E^* is locally asymptotically stable.

We also note that

$$-\frac{\partial r}{\partial U_1} = \frac{1}{(1 + r_1 U_1)^2} < 1, \qquad -\frac{\partial r}{\partial U_2} = \frac{1}{(1 + r_2 U_2)^2} < 1,$$

$$-\frac{\partial K}{\partial T_1} = \frac{1}{(1 + m_1 T_1)^2} < 1, \qquad -\frac{\partial K}{\partial T_2} = \frac{1}{(1 + m_2 T_2)^2} < 1 \ .$$

Therefore in theorem 5.2.3, we can choose

$$\rho_1 = \rho_2 = k_1 = k_2 = 1.$$

Also choose $K_m = 5.1$.

With the above set of parameters, it can further be checked that all conditions of theorem 5.2.3 are satisfied in the region

$$\mathbb{R} = \left\{ (B, T_1, T_2, U_1, U_2) : 0 \le B \le 12.3859, \ 0 \le (T_1 + T_2 + U_1 + U_2) \le 9 \right\}.$$

This shows that E^* is globally asymptotically stable.

Example 2.

In this example we choose the following function

$$r(U_1, U_2) = r_0 - r_1 U_1 - r_2 U_2 \qquad (5.42a)$$

$$K(T_1, T_2) = K_0 - K_1 T_1 - K_2 T_2 \qquad (5.42b)$$

Now we choose the following set of parameters:

$\alpha_1 = 0.1$, $\alpha_2 = 0.2$, $\beta_1 = 2$, $\beta_2 = 3$, $\delta_1 = 5$, $\delta_2 = 6$, $\nu_1 = 1$, $\nu_2 = 2$, $\pi_1 = 0.5$, $\pi_2 = 0.1$, $Q_{10} = 10$, $Q_{20} = 20$

and the functions $r(U_1, U_2)$ and $K(T_1, T_2)$ in the model (5.1) as follows.

$$r(U_1, U_2) = 4 - 0.1\ U_1 - 0.2\ U_2$$

$$K(T_1, T_2) = 10.2725 - 0.01\ T_1 - 0.02\ T_2$$

By choosing the above set of parameters in (5.1), it can be checked that the conditions (5.10) and (5.13) for the existence of $E^*(B^*, T_1^*, T_2^*, U_1^*, U_2^*)$ are satisfied and thus E^* can be found as follows.

$$B^* \simeq 10.2, \quad T_1^* \simeq 1.7878, \quad T_2^* \simeq 2.5438, \quad U_1^* \simeq 0.1495, \quad U_2^* \simeq 0.2218.$$

It can also be checked that the conditions corresponding to equations (5.19) in theorem 5.2.2 are satisfied for these set of parameters showing that E^* is locally asymptotically stable. However, with the above set of parameters it can be checked that the conditions corresponding to equations (5.21) in theorem 5.2.3 are not satisfied which shows that E^* is not globally stable.

Example 3.

In this example we choose other set of parameters and the functions $r(U_1, U_2)$, $K(T_1, T_2)$ given by (5.42) such that E^* exists and it is globally stable.

Choose the following set of parameters in the model (5.1):

$\alpha_1 = 0.01,\ \alpha_2 = 0.02,\ \beta_1 = 8,\ \beta_2 = 10,\ \delta_1 = 10,\ \delta_2 = 12,$

$\nu_1 = 0.02,\ \nu_2 = 0.03,\ \pi_1 = 0.04,\ \pi_2 = 0.05,\ Q_{10} = 10,\ Q_{20} = 20,$

$K_m = \min K(T_1, T_2) = 2.$

$r(U_1, U_2) = 10 - 0.1\, U_1 - 0.2\, U_2$

$K(T_1, T_2) = 5.545 - 0.01\, T_1 - 0.02\, T_2$

Then it can be checked that the interior equilibrium E^* exists and is given by

$$B^* \simeq 5.5,\ T_1^* \simeq 0.9945,\ T_2^* \simeq 1.6515,\ U_1^* \simeq 0.0067,\ U_2^* \simeq 0.0179.$$

It can be checked that the conditions corresponding to (5.19) in theorem 5.2.2 are satisfied and hence E^* is locally asymptotically stable.

With the values of the parameters and functions chosen above it can further be checked that the conditions corresponding to (5.21) in theorem 5.2.3 are also satisfied in the region

$$\mathbb{R} = \left\{ (B, T_1, T_2, U_1, U_2):\ 0 \le B \le 5.545,\ 0 \le \sum_{i=1}^{2} (T_i + U_i) \le 3.75 \right\}.$$

This shows that E^* is globally asymptotically stable.

Example 4.

In this example we choose the same functions given by (5.41) in which the values of parameters are given below.

$a_1 = a_2 = b_1 = b_2 = 1.0,\ r_1 = 0.2,\ r_2 = 0.1,\ m_1 = 0.02,$

$m_2 = 0.01,\ r_0 = 3.3393,\ K_0 = 14.2 .$

Now we choose the following set of parameters in model (5.34).

$\alpha_1 = 0.02,\ \alpha_2 = 0.01,\ \beta_1 = 10.0,\ \beta_2 = 12.0,\ \delta_1 = 14.0,\ \delta_2 = 15.0,$

$\nu_1 = 0.03,\ \nu_2 = 0.04,\ \pi_1 = 0.05,\ \pi_2 = 0.06,\ Q_{10} = 50.0,$

$Q_{20} = 40.0,\ \gamma = 0.2,\ \gamma_0 = 0.18,\ \gamma_1 = 1.0,\ \gamma_2 = 1.0,\ \gamma_{10} = 0.16,$

$\gamma_{20} = 0.15,\ r_{11} = 0.12,\ r_{10} = 0.11,\ r_{20} = 0.08,\ T_{c1} = 0.5,$

$T_{c2} = 0.4 .$

With the above values of parameters, it can be checked that the interior equilibrium $\hat{E}(\hat{B}, \hat{T}_1, \hat{T}_2, \hat{U}_1, \hat{U}_2, \hat{F}, \hat{F}_1, \hat{F}_2)$ exists and is given by

$$\hat{B} \simeq 8.35, \quad \hat{T}_1 \simeq 3.3892, \quad \hat{T}_2 \simeq 2.575, \quad \hat{U}_1 \simeq 0.0548,$$

$$\hat{U}_2 \simeq 0.0173, \quad \hat{F} \simeq 0.4577, \quad \hat{F}_1 \simeq 14.9325, \quad \hat{F}_2 \simeq 10.5 \ .$$

It can also be checked that the conditions (5.37) in theorem 5.4.1 are also satisfied and hence $\hat{E}$ is locally asymptotically stable.

As in the example 1, we can choose the following parameters in theorem 5.4.2 as

$$\hat{\rho}_1 = \hat{\rho}_2 = \hat{k}_1 = \hat{k}_2 = 1.0$$

Also choose $\hat{K}_m = 6.0$ in theorem 5.4.2.

Then it can be checked that all conditions in theorem 5.4.2 are satisfied. This shows that $\hat{E}$ is globally asymptotically stable.

5.6 CONCLUSIONS

In this chapter, we have presented a mathematical model for studying the simultaneous effects of two toxicants, one is more toxic than the other, emitted into the environment on a single species biological population. The cases of instantaneous spill, constant and periodic emission of each of the two toxicants have been considered and the existence of non trivial equilibrium has been proved in each case. It has been shown that in the case of instantaneous spill, the toxicants from the environment washout completely and the population may recover back to its initial carrying capacity, the duration being dependent upon the influx and washout rates of the two toxicants. In the case of constant emission of the toxicants, it has been shown that the population settles down to an equilibrium level, which is lower than its

initial (toxicant independent) carrying capacity, the magnitude of which depends upon the emission and washout rates of the two toxicants. It has also been found that in the case of uncontrolled continuous emissions of toxicants with large influx rates the population may be doomed to extinction sooner than the case of single toxicant, other parameters in the system being the same. In the case of periodic perturbation it is shown that a periodic influx of toxicants with small amplitude induces a periodic behavior in the population dynamics and the stability behavior of the system is same as the case of the constant emission of toxicants into the environment. A conservation model is also proposed the analysis of which shows that an appropriate level of species population can be maintained by conserving the forestry resource and by controlling the concentration of toxicants.

The conclusion drawn here suggests that emissions of various kinds of toxicants in the atmosphere must be controlled without delay otherwise the survival of the biological species on the planet earth will be threatened.
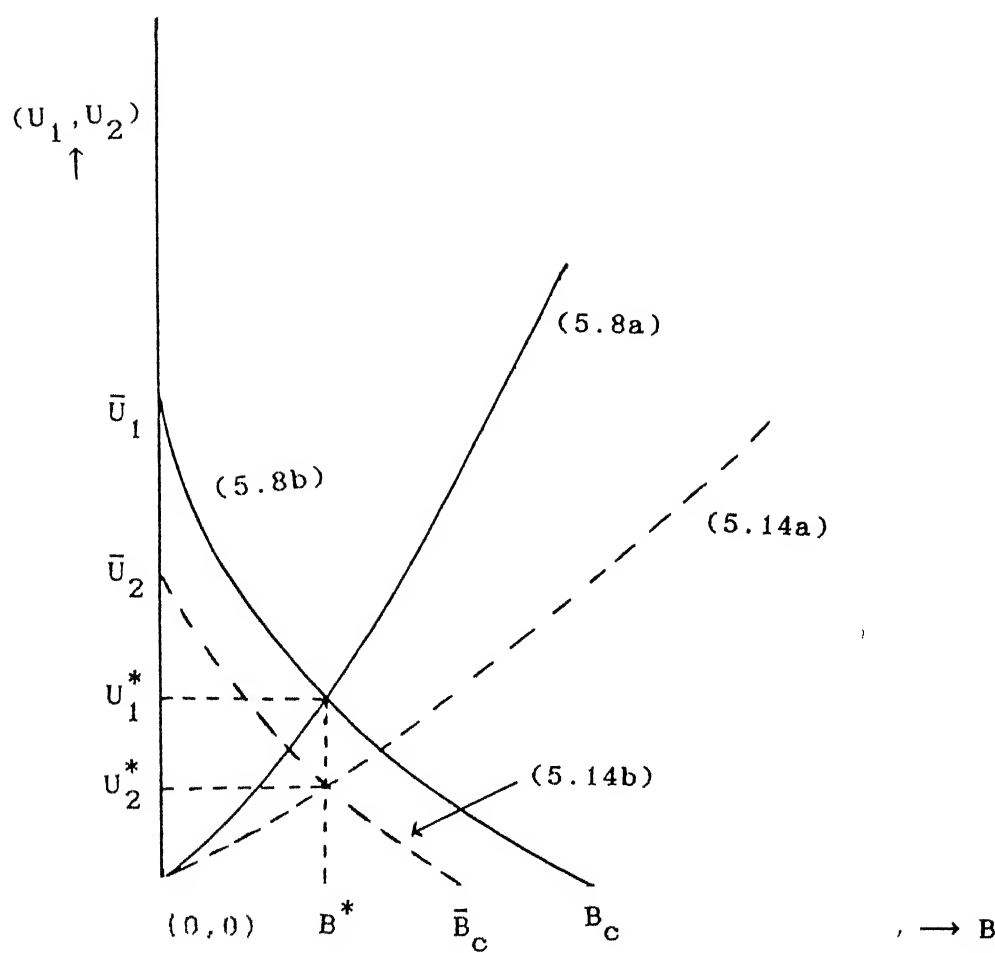
Fig. 5.1

CHAPTER VI

MODELLING THE DEPLETION AND CONSERVATION OF FERTILE TOPSOIL DEPTH: EFFECTS OF ACID RAIN AND WIND EROSION

6.0 INTRODUCTION

It is well known that fertile top soil is adversely affected by various environmental factors such as wind and storm, drought, monsoon rains and floods, acid rain, etc. leading to decrease in crop yield. A major study undertaken under the auspices of UNEP in several countries has found that among other things, in both cold regions and semi-arid zones, climate variation causing changes in wind, temperature and precipitation has adverse effect on crop yield from one region to another and from one type of soil to another, Parry and Carter(1988), Veeman(1988), Maogong(1991), Tishun and Lu Qi(1993). It is found in the North China Plain that agroforestry systems have assisted in increasing crop yields by reducing environmental stresses such as wind and storm, Tishun and Lu Qi(1993). It may be noted here that the U.S., recognizing and appreciating the importance of soil loss, started to adopt as early as in the 1930's policies and measures to control it by establishing the Soil Conservation Services which dealt with providing technical help and other information needed to farmers for soil protection, conservation and restoration. A significant achievement under this program was the natural resource inventory, a survey conducted to identify soil and water conservation needs and other related vital information. Also a Conservation Resource Program was started to control and check soil erosion and soil fertility loss by offering financial incentives to farmers and

land owners so that they may use less erodible land, provide a green cover to such land and if possible, remove such land from agricultural production related activity for a certain specified period of time.

Acid rain is one of the many environmental factors that affects the fertile topsoil leading to decrease in crop yield, Canter(1986), Howells(1990). The control of acid rain is thus an important problem all over the world.

Soil erosion due to wind velocity is another major cause of concern for the environmentalists. Since 1949, in Republic of China forestry and agroforestry practices have been used to stabilize large area of North China Plains which was subjected to violent winds. This leads to improve in agricultural production, Tishun and Lu Qi(1993).

Although some suggestions have been made to investigate the causes and consequences of topsoil erosion and the need of afforestation, Das (1977), France and Thornley (1984), Kormondy (1986), Munn and Fedorov (1986), little attention has been paid to study this problem using mathematical models, Hallam and Levin (1986), Shukla et al. (1989).

In this chapter, therefore, a mathematical model for depletion of fertile topsoil depth due to acid rain and wind is proposed and analysed. A conservation model to conserve soil depth by controlling acid rain and wind is also proposed and analysed.

6.1 MATHEMATICAL MODEL

Let us consider a habitat where we wish to model the depletion of fertile topsoil depth caused by acid rain and wind velocity. Let $S(t)$ be the depth of fertile topsoil, $C(t)$ be the

concentration of acid rain which is entering the soil with a constant rate Q and W(t) be the wind velocity at time t causing the erosion of the soil. It is assumed that the growth rate of fertile topsoil depth decreases as the concentration of acid rain and the velocity of wind increase. The decrease due to acid rain is linearly proportional to the depth of soil. However, the decrease due to wind is bilinearly proportional to the depth of soil, which may be caused due to undulation of soil surface. It is further assumed that the growth rate of acid rain decreases due to its interaction with the soil and it is proportional to the concentration of acid rain and depth of soil. Then equations governing these variables are given by

$$\frac{dS}{dt} = q - r_1 S - r_2 CS - r_0 WS^2 , \qquad S(0) = S_0 > 0$$

$$\frac{dC}{dt} = Q - m_1 C - m_2 CS , \qquad\qquad C(0) = C_0 > 0 \qquad\qquad (6.1)$$

$$\frac{dW}{dt} = \phi(t) - k_1 W , \qquad\qquad W(0) = W_0 > 0$$

where q is the natural growth rate coefficient of fertile topsoil depth assumed to be constant, r_1 is the depletion rate coefficient of soil depth due to natural factors, r_2 is the depletion rate coefficient of soil depth due to acid rain and r_0 is its depletion rate coefficient due to wind velocity W(t), m_1 is the natural depletion rate coefficients of acid rain and m_2 is the depletion rate coefficient of C(t) due to its interaction with the soil. $\phi(t)$ is the air pressure gradient causing wind which is assumed to be either constant or periodic, k_1 is the natural depletion rate coefficient of wind caused by various resistances.

6.2 MATHEMATICAL ANALYSIS

Now we analyse our model (6.1) for $q = 0$ and $q = q_0$, a positive constant.

CASE I: $q = 0$, $\phi(t) = \phi_0 > 0$

Putting $\frac{dx}{dt}$ ($x = S, C, W$) $= 0$, and solving we get only one equilibrium $E_1(0, Q/m_1, \phi_0/k_1)$. By computing the variational matrix corresponding to E_1, we note that E_1 is locally asymptotically stable. In the following theorem we also note that E_1 is globally asymptotically stable.

THEOREM 6.2.1   The equilibrium E_1 is globally asymptotically stable.

Proof: From (6.1) we note that

$$\frac{dS}{dt} < 0 \text{ and hence } \lim_{t \to \infty} S(t) = 0 \text{ for all } S \geq 0$$

Again we have

$$\frac{dC}{dt} = Q - m_1 C - m_2 CS$$

$$\leq Q - m_1 C$$

Hence $\lim_{t \to \infty} C(t) \leq Q/m_1$

Further we have

$$\frac{dW}{dt} = \phi_0 - k_1 W$$

Hence $\lim_{t \to \infty} W(t) = \phi_0/k_1$

This shows that the system is dissipative, and hence the theorem follows.

The above theorem shows that the depth of fertile topsoil always tends to zero if the concentration of acid rain and wind velocity increase without control.

CASE II: $q = q_0 > 0$, $\phi(t) = \phi_0 > 0$

In this case, the model (6.1) has again only one equilibrium $E^*(S^*, C^*, W^*)$. Here W^* is given by

$$W^* = \phi_0/k_1 \tag{6.2}$$

and S^*, C^* are the positive solution of the system of algebraic equations

$$C = \frac{Q}{m_1 + m_2 S} \tag{6.3a}$$

$$r_2 C = \frac{q_0}{S} - r_1 - \frac{r_0 \phi_0}{k_1} S \tag{6.3b}$$

From (6.3a) we note the following:

When $S \to 0$, $C \to Q/m_1$ $\qquad$ (6.4a)

When $S \to \infty$, $C \to 0$ $\qquad$ (6.4b)

and $\dfrac{dC}{dS} = - \dfrac{Q m_2}{(m_1 + m_2 S)^2} < 0 \qquad$ (6.4c)

From (6.3b) we note the following:

When $S \to 0$, $C \to \infty$ $\qquad$ (6.5a)

When $S \to S_a$, $C \to 0$ $\qquad$ (6.5b)

where S_a is given by

$$S_a = \frac{k_1}{2 r_0 \phi_0} \left[- r_1 + \left\{ r_1^2 + \frac{4 r_0 \phi_0 q_0}{k_1} \right\}^{1/2} \right] > 0 \tag{6.5c}$$

We also have from (6.3b)

$$\frac{dC}{dS} = \frac{1}{r_2} \left[- \frac{q_0}{S^2} - \frac{r_0 \phi_0}{k_1} \right] < 0 \tag{6.5d}$$

From the above analysis we note that the isoclines (6.3a) and (6.3b) intersect at a unique point [see fig. 6.1] in the positive quadrant. The intersection value of these two isoclines gives the S-C coordinates of E^*.

From (6.3a) and (6.3b), it should be noted here that S^* decreases as Q increases [see fig. 6.2].

From (6.5c) it is clear that S_a increases as q_0 increases. Let S_{a1} and S_{a2} be two values of S_a corresponding to q_{01} and q_{02}. Assume $q_{02} > q_{01}$, then $S_{a2} > S_{a1}$. Then it should be noted here that S^* increases as q_0 increases [see fig. 6.3].

From (6.5c) or putting C = 0 in (6.3b), it can also be checked that $\frac{dS_a}{d\phi_0}$ is negative. This shows that S_a decreases as ϕ_0 increases. Let S_{b1} and S_{b2} be two values of S_a corresponding to ϕ_{01} and ϕ_{02}. Assume $\phi_{02} > \phi_{01}$, then $S_{b2} < S_{b1}$. Then it can be noted here that S^* decreases as ϕ_0 increases [see fig. 6.4].

By computing the variational matrix corresponding to E^*, we can state the following theorem which can be easily proved by using Gershgorin's theorem, Lancaster and Tismanetsky(1985).

THEOREM 6.2.2 Let the following inequality holds:

$$\frac{q_0}{S^*} + \frac{r_0 \phi_0 S^*}{k_1} > m_2 C^* \qquad (6.6a)$$

$$m_1 + m_2 S^* > r_2 S^* \qquad (6.6b)$$

Then E^* is locally asymptotically stable.

In the following theorem, we have shown that E^* is also globally asymptotically stable. First we state the following lemma whose proof is similar to theorem 6.2.1 and hence omitted.

LEMMA 6.2.1 The set

$$\mathbb{R} = \left\{ (S, C, W) : 0 < S \le q_0/r_1, \ 0 < C \le Q/m_1, \ 0 < W \le \phi_0/k_1 \right\}$$

attracts all solutions initiating in the positive orthant.

We note that (6.7a) implies (6.11a) and (6.7b) implies (6.11b). Hence V is a Liapunov function with respect to E^* whose domain contains the region of attraction $\mathbb{R}$, proving the theorem.

The above theorem implies that if the concentration of acid rain and wind velocity increase without control, the depth of fertile top soil decreases and it may tend to zero even in this case if these environmental factors continue to increase without control.

CASE III: $q = q_0 > 0$, $\phi(t) = \phi_0 + \varepsilon \phi_1(t)$, $\phi_1(t + \omega) = \phi_1(t)$

In this case the system (6.1) can be written as

$$\frac{dx}{dt} = A(x) + \varepsilon B(t), \quad x(0) = x_0 \qquad (6.12)$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} S \\ C \\ W \end{bmatrix}, \quad x_0 = \begin{bmatrix} S(0) \\ C(0) \\ W(0) \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 0 \\ \varepsilon\phi_1(t) \end{bmatrix},$$

$$A(x) = \begin{bmatrix} q_0 - r_1 x_1 - r_2 x_1 x_2 - r_0 x_1^2 x_3 \\ Q - m_1 x_2 - m_2 x_1 x_2 \\ \phi_0 - k_1 x_3 \end{bmatrix}$$

Using the similar arguments as in Freedman and Shukla(1991), we can establish the following two theorems.

THEOREM 6.2.4 If M^* has no eigen value with zero real parts, then the system (6.1) with $\phi(t) = \phi_0 + \varepsilon \phi_1(t)$, $\phi_1(t + \omega) = \phi_1(t)$ has a periodic solution with period ω, $(S(t,\varepsilon), C(t,\varepsilon), W(t,\varepsilon))$ such that $(S(t,0), C(t,0), W(t,0)) = (S^*, C^*, W^*)$.

THEOREM 6.2.5 If M^* has no eigen value with zero real parts, then for sufficiently small ε, the stability behavior of the periodic solution of the system (6.1) is same as that of E^*.

6.3 CONSERVATION MODEL

To conserve the depth of fertile soil, it is necessary to control acid rain and wind velocity by using appropriate measures. Let $F_1(t)$, $F_2(t)$ and $F_3(t)$ be the densities of efforts applied to conserve the depth $S(t)$ of fertile topsoil and to control the concentration of acid rain $C(t)$ and wind velocity $W(t)$. It is assumed that $F_1(t)$ is proportional to the depleted level of fertile topsoil depth, $F_2(t)$ and $F_3(t)$ are proportional to the undesired level of the concentration of acid rain and wind velocity respectively. Then the conservation model can be written in the form of the following set of differential equations:

$$\frac{dS}{dt} = q - r_1 S - r_2 CS - r_0 WS^2 + rF_1 S , \qquad S(0) > 0$$

$$\frac{dC}{dt} = Q - m_1 C - m_2 CS - m_0 F_2, \qquad C(0) > 0$$

$$\frac{dW}{dt} = \phi(t) - k_1 W - k_2 F_3, \qquad W(0) > 0$$

$$\frac{dF_1}{dt} = \sigma(S_0 - S) - \sigma_0 F_1 , \qquad F_1(0) > 0 \qquad (6.13)$$

$$\frac{dF_2}{dt} = \nu(C - C_0) - \nu_0 F_2 , \qquad F_2(0) > 0$$

$$\frac{dF_3}{dt} = \beta(W - W_0) - \beta_0 F_3 \qquad F_3(0) > 0$$

where r is the rate of increase of fertile topsoil depth due to the effort F_1, m_0 is the depletion rate coefficient of acid rain due to the effort F_2, k_2 is the control rate coefficient of wind velocity due to the effort F_3, σ is the growth rate coefficient of effort F_1 needed to protect the fertilized soil depth, σ_0 is the natural depletion rate coefficient of effort F_1, ν is the growth rate coefficient of effort F_2 to control the acid rain and ν_0 is its natural depletion rate coefficient, β is the growth rate

coefficient of the effort F_3 to control the wind velocity $W(t)$ and β_0 is its depletion rate coefficient, S_0 is the initial density of fertile topsoil which we wish to maintain, C_0 is the concentration of acid rain and W_0 is the wind velocity harmless to the soil. We note that $S_0 > S$, $C > C_0$, $W > W_0$ for $t \geq 0$.

Now we study the stability behavior of the system (6.13) for two different values of q i.e. $q = 0$ and $q = q_0 > 0$.

CASE I: $q = 0$, $\phi(t) = \phi_0 > 0$

In this case the model (6.13) has only two nonnegative equilibria, namely $\bar{E}(0, \bar{C}, \bar{W}, \bar{F}_1, \bar{F}_2, \bar{F}_3)$ and $\widetilde{E}(\widetilde{S}, \widetilde{C}, \widetilde{W}, \widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3)$.

Here the coordinates of $\bar{E}$ are given by

$$\bar{C} = \frac{Q\nu_0 + m_0\nu C_0}{m_0\nu + m_1\nu_0} \quad , \quad \bar{W} = \frac{\phi_0\beta_0 + k_2\beta W_0}{k_1\beta_0 + k_2\beta} \quad , \quad \bar{F}_1 = \frac{\sigma S_0}{\sigma_0} \quad ,$$

$$\bar{F}_2 = \frac{\nu(Q - m_1 C_0)}{m_0\nu + m_1\nu_0} \quad , \quad \bar{F}_3 = \frac{\beta(\phi_0 - k_1 W_0)}{k_1\beta_0 + k_2\beta} \quad .$$

It should be noted here that for $\bar{F}_2$ and $\bar{F}_3$ to be positive, we must have respectively

$$Q > m_1 C_0 \quad \text{and} \quad \phi_0 > k_1 W_0 \tag{6.14}$$

The coordinates $\widetilde{S}$, $\widetilde{C}$, $\widetilde{W}$, $\widetilde{F}_1$, $\widetilde{F}_2$ and $\widetilde{F}_3$ of $\widetilde{E}$ are the positive solution of the system of algebraic equations given below.

$$r_2 C = \frac{r\sigma}{\sigma_0}(S_0 - S) - r_1 - r_0 \frac{\phi_0\beta_0 + k_2\beta W_0}{k_1\beta_0 + k_2\beta} S \quad , \tag{6.15a}$$

$$C = \frac{(Q\nu_0 + m_0\nu C_0)}{(m_0\nu + m_1\nu_0 + \nu_0 m_2 S)} \quad , \tag{6.15b}$$

$$W = \frac{\phi_0\beta_0 + k_2\beta W_0}{k_1\beta_0 + k_2\beta} = \widetilde{W} \quad , \tag{6.15c}$$

$$F_1 = \sigma(S_0 - S)/\sigma_0 \tag{6.15d}$$

$$F_2 = \nu(C - C_0)/\nu_0 \; , \tag{6.15e}$$

$$F_3 = \beta(W - W_0)/\beta_0 \; . \tag{6.15f}$$

From (6.15a) we note the following:

$$\text{When} \quad C \to C_\sigma, \quad S \to 0 \tag{6.16a}$$

$$\text{When} \quad C \to 0, \quad S \to S_\sigma \tag{6.16b}$$

where C_σ and S_σ are given by

$$C_\sigma = \frac{1}{r_2} \left[\frac{r\sigma S_0}{\sigma_0} - r_1 \right] \tag{6.16c}$$

$$S_\sigma = \frac{r\sigma S_0 - r_1 \sigma_0}{r\sigma + r_0 \sigma_0 \tilde{W}} \tag{6.16d}$$

We also have

$$\frac{dC}{dS} = \frac{1}{r_2} \left[- \frac{r\sigma}{\sigma_0} - r_0 \tilde{W} \right] < 0 \tag{6.16e}$$

From (6.15b) we note the following:

$$\text{When} \quad S \to 0, \quad C \to C_s = \frac{Q\nu_0 + m_0 \nu C_0}{m_0 \nu + m_1 \nu_0} \tag{6.17a}$$

$$\text{When } S \to \infty, \quad C \to 0 \tag{6.17b}$$

$$\text{and} \quad \frac{dC}{dS} = - \frac{\nu_0 m_0 (Q\nu_0 + m_0 \nu C_0)}{(m_0 \nu + m_1 \nu_0 + \nu_0 m_2 S)^2} < 0 \tag{6.17c}$$

From the above analysis we note that the isoclines (6.15a) and (6.15b) intersect at a unique point [see fig. 6.5] iff

$$\tag{6.18}$$

$$C_\sigma > C_s$$

The intersection value of these two isoclines gives the C-S coordinates of $\tilde{E}$ and then its other coordinates can be computed from (6.15c)-(6.15f).

Again it should be noted here that for $\tilde{F}_1$, $\tilde{F}_2$ and $\tilde{F}_3$ to be positive we must have respectively

$$S_0 > \tilde{S} , \quad \tilde{C} > C_0 \text{ and } \phi_0 > k_1 W_0 \tag{6.19}$$

By computing the variational matrices corresponding to $\bar{E}$, one can see that the eigen value in S direction is $r_2(C_\sigma - C_s)$ which is positive in view of (6.18) and hence $\bar{E}$ is locally unstable in S direction.

The following theorem gives the criteria for $\tilde{E}$ to be locally asymptotically stable which can be proved by computing variational matrix corresponding to $\tilde{E}$ and by using Gershgorin's theorem, Lancaster and Tismanetsky (1985).

THEOREM 6.3.1   Let the following inequalities hold

$$r_0 \tilde{W} \tilde{S} > \sigma + m_2 \tilde{C} \tag{6.20a}$$

$$m_1 + m_2 \tilde{S} > r_2 \tilde{S} + \nu \tag{6.20b}$$

$$k_1 > \beta + r_0 \tilde{S}^2 \tag{6.20c}$$

$$\sigma_0 > r\tilde{S} \tag{6.20d}$$

$$\nu_0 > m_0 \tag{6.20e}$$

$$\beta_0 > k_2 \tag{6.20f}$$

Then $\tilde{E}$ is locally asymptotically stable.

To prove the global stability of $\tilde{E}$, we first need the following lemma which establishes the region of attraction for the system (6.13) whose proof is similar to theorem 6.2.1 and hence is omitted.

LEMMA 6.3.1   The set
$$\tilde{R} = \left\{ (S,C,W,F_1,F_2,F_3): 0 < S < \infty , \ 0 < C \leq Q/m_1, \ 0 < W \leq \phi_0/k_1, \right.$$
$$\left. 0 < F_1 \leq \sigma S_0/\sigma_0, \ 0 < F_2 \leq \nu Q/\nu_0 m_1, \ 0 < F_3 \leq \beta\phi_0/\beta_0 k_1 \right\} \text{ attracts}$$

all solutions initiating in the positive orthant.

THEOREM 6.3.2 Let the following inequalities hold

$$\left[\frac{m_2 Q}{m_1} + r_2 \right]^2 < \frac{2}{3} r_0 \tilde{W}(m_1 + m_2 \tilde{S}) \tag{6.21a}$$

$$r_0 S_m^2 < \frac{2}{3} k_1 \tilde{W} \tag{6.21b}$$

where S_m is the maximum value of S in the region $\tilde{R}$.

Then $\tilde{E}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

Proof: Consider the following positive definite function about $\tilde{E}$,

$$U = [S - \tilde{S} - \tilde{S} \ln(S/\tilde{S})] + \frac{1}{2}(C - \tilde{C})^2 + \frac{1}{2}(W - \tilde{W})^2$$

$$+ \frac{1}{2} c_1 (F_1 - \tilde{F}_1)^2 + \frac{1}{2} c_2 (F_2 - \tilde{F}_2)^2 + \frac{1}{2} c_3 (F_3 - \tilde{F}_3)^2 \tag{6.22}$$

Differentiating U with respect to t along the solution of (6.13) we get

$$\frac{dU}{dt} = - r_0 \tilde{W}(S - \tilde{S})^2 - (m_1 + m_2 \tilde{S})(C - \tilde{C})^2 - k_1(W - \tilde{W})^2$$

$$- c_1 \sigma_0 (F_1 - \tilde{F}_1)^2 - c_2 \nu_0 (F_2 - \tilde{F}_2)^2 - c_3 \beta_0 (F_3 - \tilde{F}_3)^2$$

$$+ (S - \tilde{S})(C - \tilde{C})[- r_2 - m_2 C] + (S - \tilde{S})(W - \tilde{W})[- r_0 S]$$

$$+ (S - \tilde{S})(F_1 - \tilde{F}_1)[r - c_1 \sigma]$$

$$+ (C - \tilde{C})(F_2 - \tilde{F}_2)[c_2 \nu - m_0]$$

$$+ (W - \tilde{W})(F_3 - \tilde{F}_3)[c_3 \beta - k_2] \tag{6.23}$$

which can further be written as

$$\frac{dU}{dt} = - \frac{1}{2} b_{11}(S - \tilde{S})^2 + b_{12}(S - \tilde{S})(C - \tilde{C}) - \frac{1}{2} b_{22}(C - \tilde{C})^2$$

$$- \frac{1}{2} b_{11}(S - \tilde{S})^2 + b_{13}(S - \tilde{S})(W - \tilde{W}) - \frac{1}{2} b_{33}(W - \tilde{W})^2$$

$$- \frac{1}{2} b_{11}(S - \tilde{S})^2 + b_{14}(S - \tilde{S})(F_1 - \tilde{F}_1) - \frac{1}{2} b_{44}(F_1 - \tilde{F}_1)^2$$

$$- \frac{1}{2} b_{22}(C - \tilde{C})^2 + b_{25}(C - \tilde{C})(F_2 - \tilde{F}_2) - \frac{1}{2} b_{55}(F_2 - \tilde{F}_2)^2$$

$$- \frac{1}{2} b_{33}(W - \tilde{W})^2 + b_{36}(W - \tilde{W})(F_3 - \tilde{F}_3) - \frac{1}{2} b_{66}(F_3 - \tilde{F}_3)^2 \tag{6.24}$$

where $b_{11} = \frac{2}{3} r_0 \tilde{W}$, $b_{22} = (m_1 + m_2 \tilde{S})$, $b_{33} = k_1$,

$\qquad$ $b_{44} = 2c_1 \sigma_0$, $b_{55} = 2c_2 \nu_0$, $b_{66} = 2c_3 \beta_0$,

$\qquad$ $b_{12} = - r_2 - m_2 C$, $b_{13} = - r_0 S$, $b_{14} = r - c_1 \sigma$,

$\qquad$ $b_{25} = c_2 \nu - m_0$, $b_{36} = c_3 \beta - k_2$

Then the sufficient conditions for $\frac{dU}{dt}$ to be negative definite are that the following inequalities hold:

$$b_{12}^2 < b_{11} b_{22} \qquad (6.25a)$$

$$b_{13}^2 < b_{11} b_{33} \qquad (6.25b)$$

$$b_{14}^2 < b_{11} b_{44} \qquad (6.25c)$$

$$b_{25}^2 < b_{22} b_{55} \qquad (6.25d)$$

$$b_{36}^2 < b_{33} b_{66} \qquad (6.25e)$$

Choosing $c_1 = r/\sigma$, $c_2 = m_0/\nu$, $c_3 = k_2/\beta$, we note that the inequalities (6.25c,d,e) are automatically satisfied. We also note that (6.21a) implies (6.25a) and (6.21b) implies (6.25b). Hence U is a Liapunov function with respect to $\tilde{E}$ whose domain contains the region of attraction $\tilde{R}$, proving the theorem.

The above theorem implies that the depth of the fertile topsoil can be maintained at a desired level by conserving the fertile top soil and by controlling the environmental factors.

CASE II: $q = q_0 > 0$, $\phi(t) = \phi_0 > 0$.

In this case, the model (6.13) has only one equilibrium $\hat{E}(\hat{S}, \hat{C}, \hat{W}, \hat{F}_1, \hat{F}_2, \hat{F}_3)$. Here $\hat{S}$, $\hat{C}$, $\hat{W}$, $\hat{F}_1$, $\hat{F}_2$ and $\hat{F}_3$ are the positive solution of the following set of equations:

$$r_2 C = \frac{q}{S} + \frac{r\sigma}{\sigma_0} (S_0 - S) - r_1 - r_0 \frac{\phi_0 \beta_0 + k_2 \beta W_0}{k_1 \beta_0 + k_2 \beta} S \; , \qquad (6.26a)$$

$$C = \frac{(Q\nu_0 + m_0 \nu C_0)}{(m_0 \nu + m_1 \nu_0 + \nu_0 m_2 S)} \; , \qquad (6.26b)$$

$$W = \frac{\phi_0 \beta_0 + k_2 \beta W_0}{k_1 \beta_0 + k_2 \beta} = \hat{W} \; , \qquad (6.26c)$$

$$F_1 = \frac{\sigma(S_0 - S)}{\sigma_0 + \sigma_1 S} \; , \qquad (6.26d)$$

$$F_2 = \nu(C - C_0)/\nu_0 \; , \qquad (6.26e)$$

$$F_3 = \beta(W - W_0)/\beta_0 \; . \qquad (6.26f)$$

The analysis of isocline (6.26b) is same as that of isocline (6.15b). From (6.26a) we note the following:

When $S \to 0$, $C \to \infty$ $\qquad (6.27a)$

When $C \to 0$, $S \to S_c$ $\qquad (6.27b)$

where S_c is given by

$$S_c = \frac{1}{2r_0 \sigma_0 \hat{W}} \left[- (r\sigma + r_1 \sigma_1) + \left\{ (r\sigma + r_1 \sigma_1)^2 + 4 r_0 \sigma_0 W (q_0 \sigma_0 + r\sigma S_0) \right\}^{1/2} \right]$$

$$(6.27c)$$

From (6.26a) it is also clear that $\frac{dC}{dS}$ is negative.

Thus the isoclines (6.26a) and (6.26b) must intersect at a unique point in the positive quadrant. The intersection value of these two isoclines gives the S-C coordinates of $\hat{E}$ and its other coordinates can be computed from (6.26c)-(6.26f). It should be noted here that for $\hat{F}_1$, $\hat{F}_2$ and $\hat{F}_3$ to be positive, we must have respectively

$$S_0 > \hat{S}, \quad \hat{C} > C_0 \text{ and } \phi_0 > k_1 W_0 \qquad (6.28)$$

The following theorem gives the criteria for $\hat{E}$ to be locally stable whose proof is similar to theorem 6.3.1 and hence is omitted.

THEOREM 6 3.3   Let the following inequalities hold

$$\frac{q}{\hat{S}} + r_0 \hat{W}\hat{S} > \sigma + m_2 \hat{C} \tag{6.29a}$$

$$m_1 + m_2 \hat{S} > r_2 \hat{S} + \nu \tag{6.29b}$$

$$k_1 > \beta + r_0 \hat{S}^2 \tag{6.29c}$$

$$\sigma_0 > r\hat{S} \tag{6.29d}$$

$$\nu_0 > m_0 \tag{6.29e}$$

$$\beta_0 > k_2 \tag{6.29f}$$

Then $\hat{E}$ is locally asymptotically stable.

In this case, the region of attraction for the system (6.13) is same as $\tilde{\mathbb{R}}$.

In the following theorem we have shown that $\hat{E}$ is globally asymptotically stable whose proof is similar to theorem 6.3.2 and hence we omit it.

THEOREM 6.3.4   Let the following inequalities hold

$$\left[\frac{m_2 Q}{m_1} + r_2 \right]^2 < \frac{2}{3} r_0 \hat{W}(m_1 + m_2 \hat{S}) \tag{6.30a}$$

$$r_0 \hat{S}_m^2 < \frac{2}{3} k_1 \hat{W} \tag{6.30b}$$

where $\hat{S}_m$ is the maximum value of S in the region $\tilde{\mathbb{R}}$.

Then $\hat{E}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

The above theorem implies that if suitable measures to control acid rain and wind velocity are taken, an appropriate level of fertile soil depth can be maintained.

REMARK: If $\phi(t)$ is periodic, then the analysis is similar to the case III of the model (6.1) and hence omitted.

6.4 CONCLUSIONS

In this chapter the effect of environmental factors such as acid rain and wind on the depletion of the fertile topsoil is investigated. It is assumed that the growth rate of fertile topsoil is zero or a positive constant. It is further assumed that the pressure gradient of wind is either constant or periodic. In modelling the system, it has been considered that the growth rate of fertile topsoil decreases as the concentration of acid rain and wind velocity increase. The decrease in the growth rate of fertile topsoil due to acid rain is proportional to the depth of fertile soil. But the decrease in the growth rate of fertile topsoil due to wind velocity is bilinearly proportional to the depth of fertile soil which may be due to undulation of soil surface. It is also assumed that the growth rate of acid rain decreases due ti its interaction with the soil and it is proportional to the concentration of acid rain and depth of soil.

When the growth rate of topsoil depth is zero, it is shown that as the concentration of acid rain increases or wind velocity increases the depth of fertile topsoil decreases and always tends to zero under sustained increase of these factors. When the growth rate of topsoil depth is a constant, it is shown that the increase in the magnitude of environmental factors will lower the depth of fertile topsoil and it may tend to zero even in this case if these factors continue to increase. When the pressure gradient of wind is periodic with small amplitude, it is shown that a periodic behavior occurs in the system and its stability behavior is same as that of the constant pressure gradient of wind.

A model to conserve the topsoil and to control acid rain and wind is also proposed and analysed. It is assumed here that the density of the effort applied to conserve the topsoil is proportional to the depleted depth of topsoil and the densities of the efforts applied to control the concentration of acid rain and wind velocity are proportional to their respective undesired levels. By analysing the model it is shown that if suitable efforts are applied to conserve the topsoil depth and to control these environmental factors, an appropriate level of fertile topsoil depth can be maintained.
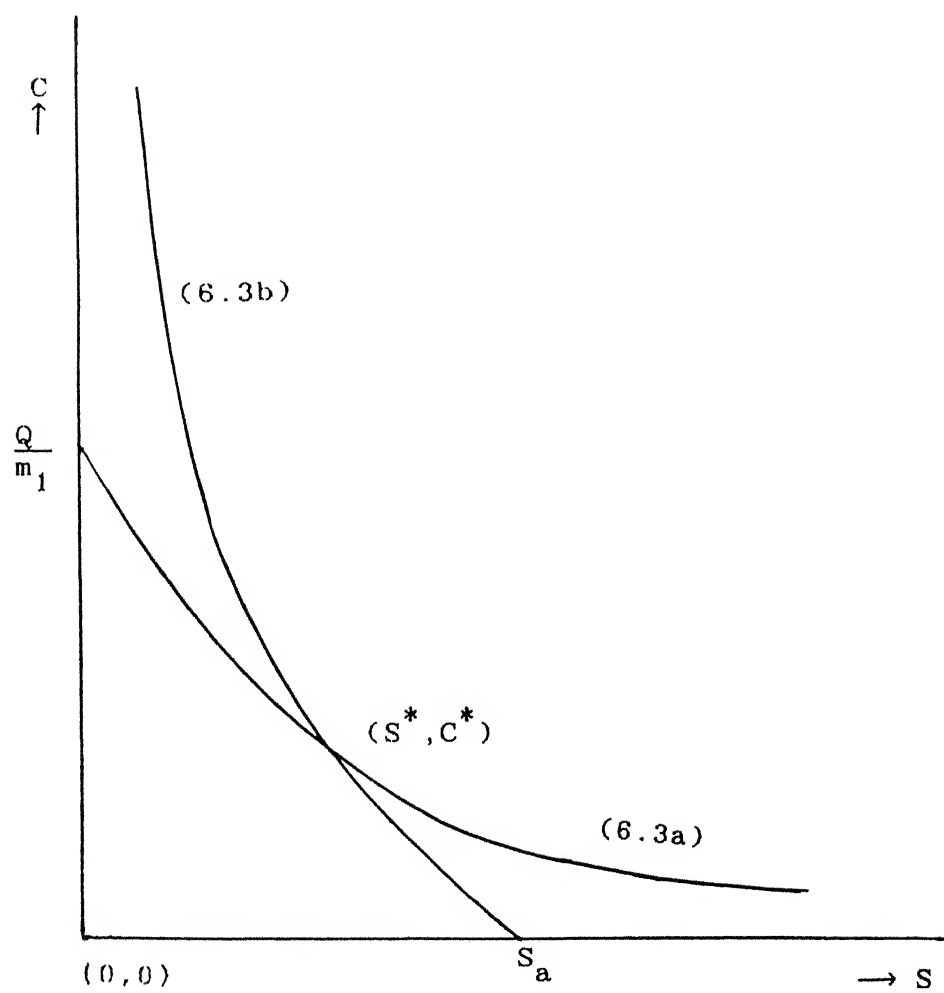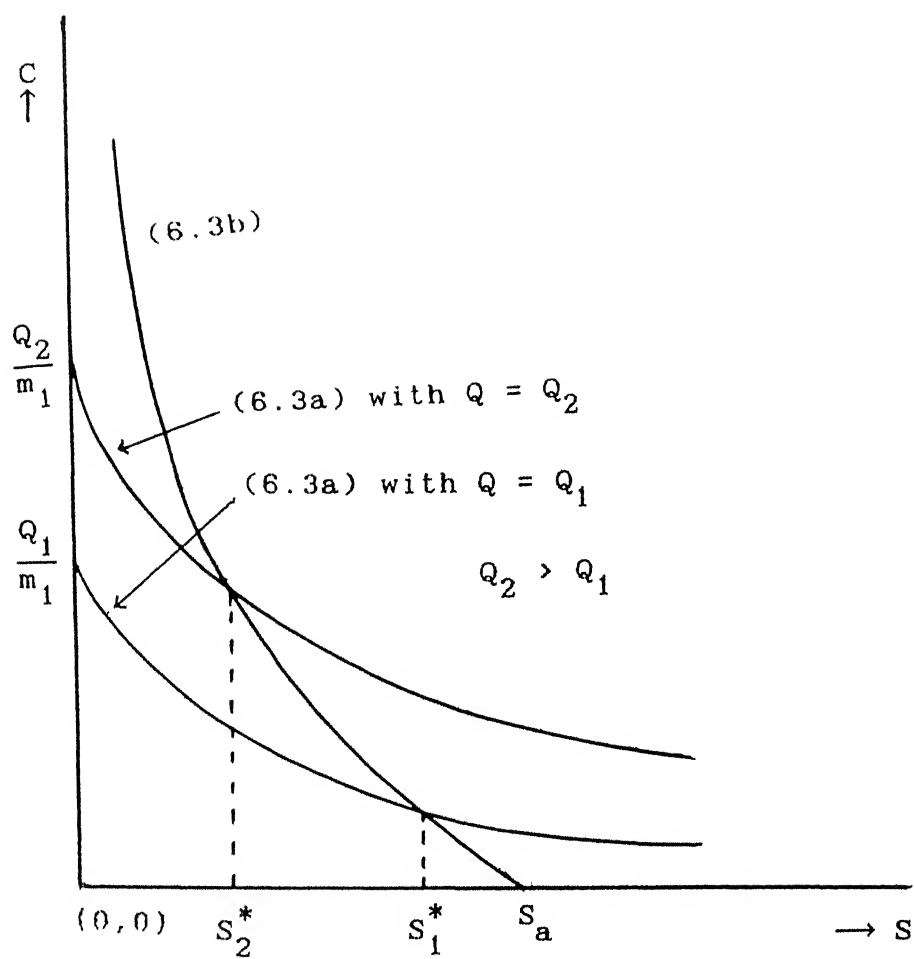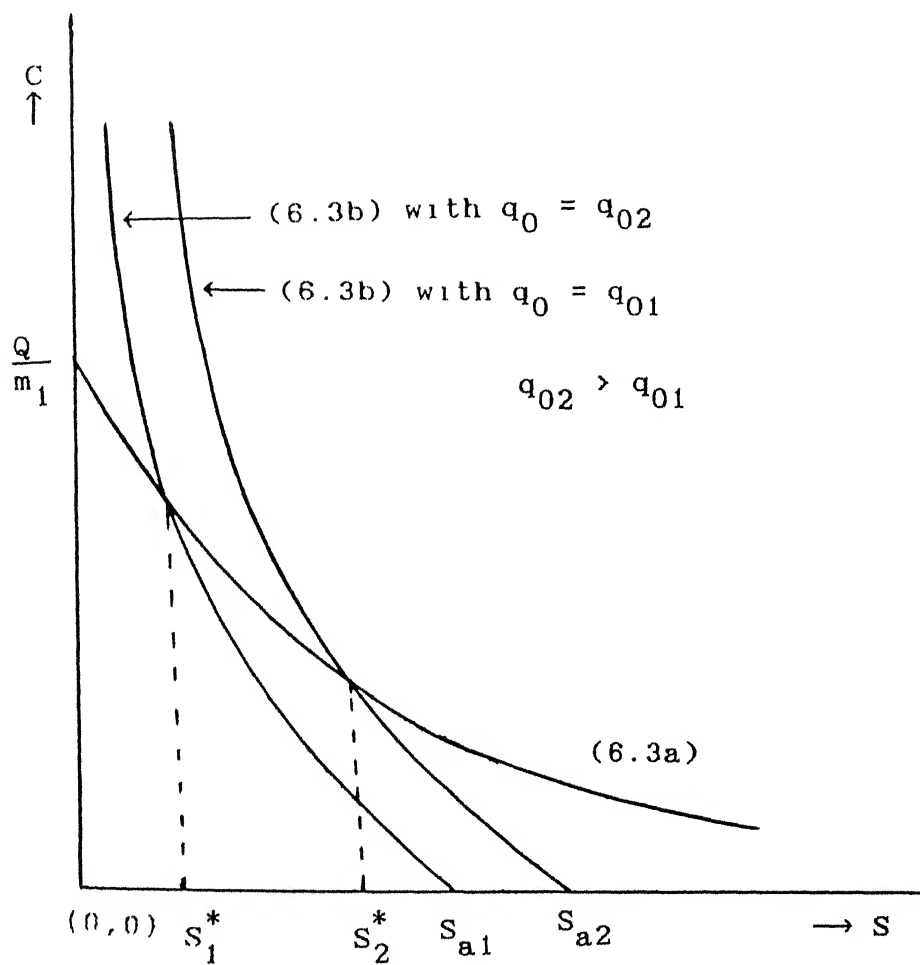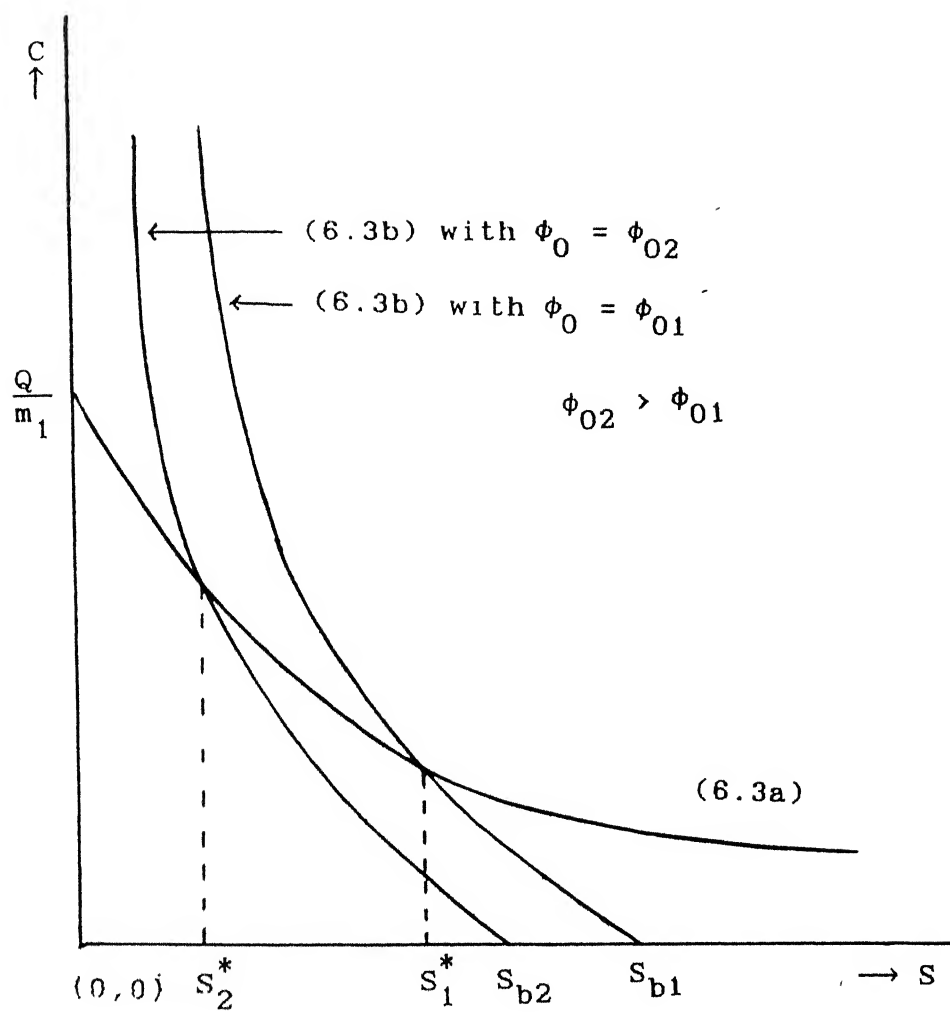
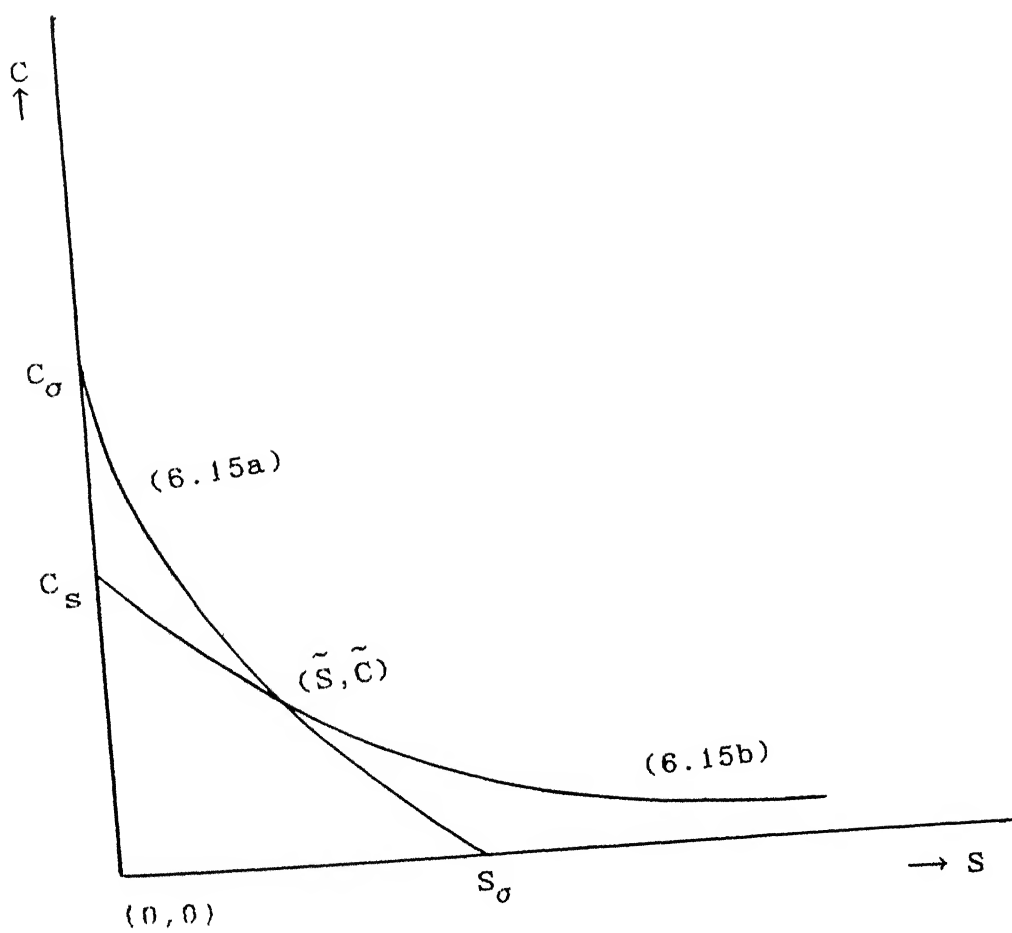156

Fig. 6.1

Fig. 6.2

Fig. 6.3

Fig. 6.4

Fig. 6.5

CHAPTER VII

MODELLING THE DEPLETION AND CONSERVATION OF FORESTRY RESOURCES: EFFECTS OF TWO INTERACTING POPULATIONS

7.0 INTRODUCTION

One of the important problem in mathematical ecology is to understand the effects of various types of industrialization and overgrowing populations on the resource utilization all around the world. A typical example in this regard is the case of forest biomass depletion and forest land use in the Doon Valley of Uttar Pradesh in India due to increase of human and cattle populations and due to increase of logging industries, Munn and Fedorov (1986), Shukla et al. (1989).

A very few investigations have been made to model the use of resources by populations, Hollings (1965), Goh (1976), Harrison (1979), Hsu and Hubbel (1979), Hsu (1981a), Freedman and So (1985), Gopalsamy (1986), Shukla et al. (1988), Shukla et al. (1989), Freedman and Shukla (1989). In particular, Goh(1976) found sufficient conditions for global stability in two species interaction but did not consider the effects of resource in the model. In this regard Hsu(1981a) developed a resource based competition model with interference. Gopalsamy(1986) also proposed a resource based competition model and found sufficient conditions for steady state to be globally asymptotically stable. Shukla et al. (1989)considered the cumulative effect of population and industrialization, growing logistically, on resource depletion and have shown that resource may vanish if these factors grow without control.

It may be noted here that in these investigations the effect of resource biomass on the growth rate and the corresponding carrying capacity of populations as well as the interaction between themselves are not considered.

Keeping in view the above, in this chapter a general model is proposed for the use of a resource by two interacting populations. Four types of interactions between the populations are considered, namely (i) two populations are competing for the resource, (ii) one population is an alternative resource for the other, (iii) one population is a prey for the other predator population and (iv) two populations are cooperating each other. It is assumed that the growth rate density of resource as well as its carrying capacity decrease as the density of populations increases. It is also assumed that the growth rate and carrying capacity of populations increase with the density of resource biomass.

Stability theory (La Salle and Lefschetz(1961)) is used for model analysis. A prime, where there is no confusion, denotes the derivative of a function with respect to its argument. 7.1   7.1

7.1 MATHEMATICAL MODEL

We consider an ecosystem where we wish to model the effects of two interacting populations on the use of a resource. We assume that the dynamics of resource biomass and populations are governed by generalized logistic equations. We assume that the growth rate of resource biomass as well as its carrying capacity decreases as the density of populations increases. We further assume that the growth rate and carrying capacity of both of the populations increase with the density of resource biomass. Then the dynamics of the system can be governed by means of the following autonomous

differential equations :

$$\frac{dB}{dt} = r_B(N_1,N_2)B - \frac{r_{B0}B^2}{K_B(N_1,N_2)}$$

$$\frac{dN_1}{dt} = r_1(B)N_1 - \frac{r_{10}N_1^2}{K_1(B)} - \alpha N_1 N_2 \qquad (7.1)$$

$$\frac{dN_2}{dt} = r_2(B)N_2 - \frac{r_{20}N_2^2}{K_2(B)} - \beta N_1 N_2$$

$$B(0) \geq 0, \ N_1(0) \geq 0, \ N_2(0) \geq 0.$$

Here $B(t)$ is the density of resource biomass, $N_1(t)$ and $N_2(t)$ are the densities of two interacting populations at time t.

The constants α and β are interspecific interference coefficients of populations which may be positive, negative or zero depending upon the nature of their interactions.

CASE I: Competition model

In this case we assume that α and β are positive constants and they denote the measure of damage effect on the competitors. Here, the two populations are competing for the resource biomass.

CASE II: Alternative resource model

In this case we assume that α and β are of opposite sign, say $\alpha = a_{12}$, $\beta = -a_{21}$; $a_{12} > 0$, $a_{21} > 0$. Here the population of density $N_1(t)$ is a supplementary food for the population of density $N_2(t)$ due to which growth rate of the first population decreases with the rate coefficient a_{12} and growth rate of the second population increases with the rate coefficient a_{21}.

CASE III: Predation type model

In this case we assume $\alpha = a_{12} > 0$, $\beta = -a_{21} < 0$ and $r_2(0) = -r_{20} < 0$, $r_2(B_c) = 0$ for some $B_c > 0$, $r_2'(B) > 0$ for $B \geq 0$ then in the model (8.1) N_1 acts as a predator and N_2 as a prey.

CASE IV: Cooperation model

In this case we assume that α and β are negative, say $\alpha = -b_{12}$, $\beta = -b_{21}$; $b_{12} > 0$, $b_{21} > 0$. Here the two populations are cooperating each other and hence the growth rate of each population increases due to the presence of the other.

In the model (7.1), the function $r_B(N_1, N_2)$ denotes the specific growth rate of resource biomass which decreases as N_1 and N_2 increase and hence we assume

$$r_B(0,0) = r_{B0} > 0, \quad \frac{\partial r_B(N_1, N_2)}{\partial N_1} < 0, \quad \frac{\partial r_B(N_1, N_2)}{\partial N_2} < 0 \text{ for } N_1 \geq 0, N_2 \geq 0$$

and $r_B(\bar{N}_1, 0) = r(0, \bar{N}_2) = 0$ for some $\bar{N}_1 > 0$, $\bar{N}_2 > 0$. (7.2)

The function $K_B(N_1, N_2)$ may be interpreted as the maximum density of the resource biomass which the environment can support in presence of the two populations of density N_1 and N_2 and it decreases as N_1 and N_2 increase. Hence we assume

$$K_B(0,0) = K_{B0} > 0, \quad \frac{\partial K_B(N_1, N_2)}{\partial N_1} < 0, \quad \frac{\partial K_B(N_1, N_2)}{\partial N_2} < 0 \text{ for } N_1 \geq 0, N_2 \geq 0$$

(7.3)

The function $K_i(B)$ may represent the maximum density of i-th population which the environment can support in presence of the alternative resource biomass of density B and it increases with the density of biomass. Hence we assume

$$K_i(0) = K_{i0} > 0, \quad K_i'(B) > 0 \text{ for } B \geq 0, i = 1, 2. \tag{7.4}$$

The function $r_i(B)$ represents the specific growth rate of i-th population which increases with B and hence we assume

$$r_i(0) = r_{i0} > 0, \quad r_i'(B) > 0 \text{ for } B \geq 0, i = 1, 2. \tag{7.5}$$

REMARK   In (7.5), $r_i(0)$ is taken positive. However, $r_i(0)$ may be zero or negative. If $r_1(0) = 0$, then the two populations wholly depend upon the resource. If $r_i(0) = -r_{i0} < 0$, then the two populations are predating on the resource. In this case the model can also be applied to the depletion of forestry resource by two kinds of industries such as paper and furniture.

7.2   MATHEMATICAL ANALYSIS

7.2.1a   THE CASE OF TWO COMPETING SPECIES WITH ALTERNATIVE RESOURCE.

In this case $\alpha < 0$, $\beta > 0$ and $r_i(0) = r_{i0} > 0$.

Then the equilibria of the model (7.1) are given by

$E_0(0,0,0)$, $E_1(K_{B0},0,0)$, $E_2(0,K_{10},0)$, $E_3(0,0,K_{20})$, $E_4(B_a^*,N_{1a}^*,0)$, $E_5(B_b^*,0,N_{2b}^*)$, $E_6(0,N_{1c}^*,N_{2c}^*)$ and $E^*(B^*,N_1^*,N_2^*)$. The existence of equilibria E_0 to E_3 is obvious. We shall show the existence of other equilibria as follows.

Existence of $E_4(B_a^*,N_{1a}^*,0)$:

Here B_a^* and N_{1a}^* are the positive solution of the system of algebraic equations

$$r_{B0}B = r_B(N_1,0)K_B(N_1,0) \tag{7.6a}$$

$$r_1(B)K_1(B) = r_{10}N_1 \tag{7.6b}$$

It is easy to check that the isoclines (7.6a) and (7.6b) intersect at a unique point (B_a^*,N_{1a}^*) iff

$$K_{10} < \bar{N}_1 \tag{7.7}$$

Thus, the equilibrium E_4 exists under the condition (7.7).

Existence of $E_5(B_b^*,0,N_{2b}^*)$:

Here B_b^* and N_{2b}^* are the positive solution of the system of

algebraic equations

$$r_{B0}B = r_B(0,N_2)K_B(0,N_2) \tag{7.8a}$$

$$r_2(B)K_2(B) = r_{20}N_2 \tag{7.8b}$$

As in the existence of E_4, one can see that the equilibrium E_5 exists iff

$$K_{20} < \bar{N}_2 \tag{7.9}$$

Existence of $E_6(0,N_{1c}^*,N_{2c}^*)$:

Here N_{1c}^* and N_{2c}^* are given by

$$N_{1c}^* = \frac{r_{20}K_{10}}{\delta_0} (r_{10} - \alpha K_{20}) \tag{7.10a}$$

$$N_{2c}^* = \frac{r_{10}K_{20}}{\delta_0} (r_{20} - \beta K_{10}) \tag{7.10b}$$

where $\delta_0 = r_{10}r_{20} - \alpha\beta K_{10}K_{20}$ $\qquad$ (7.10c)

It is clear that the equilibrium E_6 exists iff

$$\text{either (i)} \quad r_{10} > \alpha K_{20}, \quad r_{20} > \beta K_{10} \tag{7.11a}$$

$$\text{or} \qquad \text{(ii)} \quad r_{10} < \alpha K_{20}, \quad r_{20} < \beta K_{10} \tag{7.11b}$$

holds i. e. E_6 exists iff $\delta_0 \neq 0$.

Existence of interior equilibrium $E^*(B^*,N_1^*,N_2^*)$:

Here B^*, N_1^* and N_2^* are the positive solution of the system of algebraic equations

$$r_{B0}B = r_B(N_1,N_2)K_B(N_1,N_2) \tag{7.12a}$$

$$N_1 = \frac{K_1(B)[r_{20}r_1(B) - \alpha r_2(B)K_2(B)]}{r_{10}r_{20} - \alpha\beta K_1(B)K_2(B)} = f(B) , \quad \text{(say)} \tag{7.12b}$$

$$N_2 = \frac{K_2(B)[r_{10}r_2(B) - \beta r_1(B)K_1(B)]}{r_{10}r_{20} - \alpha\beta K_1(B)K_2(B)} = g(B) , \quad \text{(say)} \tag{7.12c}$$

Since α, β are coefficients of competition, we note from (7.12b)

that N_1 increases as β increases keeping other functions in (7.12b) constant and from (7.12c) we note that N_2 increases as α increases keeping other functions in (7.12c) constant.

Substituting N_2 from (7.12c) in (7.12a) we get

$$r_{BO}B = r_B(N_1,g(B))K_B(N_1,g(B)) \qquad (7.13)$$

Now we shall show that the isocline (7.12b) and (7.13) intersect at a unique point.

From (7.12b) we note the following:

When $B \rightarrow 0$, $N_1 \rightarrow f(0) = N_{1c}^*$

where N_{1c}^* is given by (7.10a) and it is positive under the condition (7.11).

It is clear that $f'(B)$ may be positive or negative depending upon the value of the functions in (7.12b). If we assume

$$f'(B) > 0 \text{ for } B \geq 0 \qquad (7.14a)$$

then in (7.12b), N_1 is an increasing function of B starting from N_{1c}^*.

From (7.13) we note the following:

When $N_1 \rightarrow 0$, $B \rightarrow B_k^*$

where B_k^* is a zero of

$$F(B) = r_{BO}B - r_B(0,g(B))K_B(0,g(B))$$

We see that $F(0) < 0$ for $g(0) = N_{2c}^* < \bar{N}_2$ and $F(K_{BO}) > 0$, hence there exists a B_k^* in the interval $0 < B_k^* < K_{BO}$ such that $F(B_k^*) = 0$ Further we have

When $B \rightarrow 0$, $N_1 \rightarrow N_{10}^*$

where N_{10}^* in the interval $0 < N_{10}^* < \bar{N}_1$ satisfies $r_B(N_{10}^*,g(0)) = 0$.

We also have from (7.13)

$$\frac{dN_1}{dB} \left[r_B(N_1, g(B)) \frac{\partial K_B}{\partial N_1} + K_B(N_1, g(B)) \frac{\partial r_B}{\partial N_1} \right]$$

$$= r_{B0} - r_B(N_1, g(B)) \frac{\partial K_B}{\partial N_2} \frac{dg}{dB} - K_B(N_1, g(B)) \frac{\partial r_B}{\partial N_2} \frac{dg}{dB}$$

which shows that $\frac{dN_1}{dB} < 0$, provided

$$g'(B) > 0 \text{ for } B \geq 0 \qquad (7.14b)$$

Thus from the above analysis, we note that the isoclines (7.12b) and (7.13) intersect at a unique point (B^*, N_1^*) [see fig.7.1], provided

$$N_{1c}^* < N_{10}^* \qquad (7.14c)$$

Knowing the values of B^* and N_1^*, N_2^* can be calculated from (7.12c). It should be noted that for N_1^* and N_2^* to be positive

either (i) $r_{20}r_1(B^*) > \alpha r_2(B^*)K_2(B^*)$,

$$r_{10}r_2(B^*) > \beta r_1(B^*)K_1(B^*), \qquad (7.15a)$$

or (ii) $r_{20}r_1(B^*) < \alpha r_2(B^*)K_2(B^*)$,

$$r_{10}r_2(B^*) < \beta r_1(B^*)K_1(B^*) \qquad (7.15b)$$

must be satisfied.

In this case to study the stability behavior of the equilibria of the model (7.1) we compute the variational matrices corresponding to each equilibrium as follows.

$$M_0 = \begin{bmatrix} r_{B0} & 0 & 0 \\ 0 & r_{10} & 0 \\ 0 & 0 & r_{20} \end{bmatrix}$$

$$M_1 = \begin{bmatrix} -r_{BO} & E_{12} & E_{13} \\ 0 & r_1(K_{BO}) & 0 \\ 0 & 0 & r_2(K_{BO}) \end{bmatrix}$$

where

$$E_{12} = K_{BO}\frac{\partial r_B(0,0)}{\partial N_1} + r_{BO}\frac{\partial K_B(0,0)}{\partial N_1} \ , \qquad (7.16a)$$

$$E_{13} = K_{BO}\frac{\partial r_B(0,0)}{\partial N_2} + r_{BO}\frac{\partial K_B(0,0)}{\partial N_2} \ . \qquad (7.16b)$$

$$M_2 = \begin{bmatrix} r_B(K_{10},0) & 0 & 0 \\ r_{10}K_1'(0) + K_{10}r_1'(0) & -r_{10} & -\alpha K_{10} \\ 0 & 0 & r_{20} - \beta K_{10} \end{bmatrix}$$

$$M_3 = \begin{bmatrix} r_B(0,K_{20}) & 0 & 0 \\ 0 & r_{10} - \alpha K_{20} & 0 \\ r_{20}K_2'(0) + K_{20}r_2'(0) & -\beta K_{20} & -r_{20} \end{bmatrix}$$

$$M_4 = \begin{bmatrix} -r_B(N_{1a}^*,0) & F_{12} & F_{13} \\ F_{21} & -r_1(B_a^*) & -\alpha N_{1a}^* \\ 0 & 0 & r_2(B_a^*) - \beta N_{1a}^* \end{bmatrix}$$

where

$$F_{12} = B_a^* \frac{\partial r_B(N_{1a}^*, 0)}{\partial N_1} + \frac{r_B^2(N_{1a}^*, 0)}{r_{BO}} \frac{\partial K_B(N_{1a}^*, 0)}{\partial N_1} \qquad (7.17a)$$

$$F_{13} = B_a^* \frac{\partial r_B(N_{1a}^*, 0)}{\partial N_2} + \frac{r_B^2(N_{1a}^*, 0)}{r_{BO}} \frac{\partial K_B(N_{1a}^*, 0)}{\partial N_2} \qquad (7.17b)$$

$$F_{23} = r_1'(B_a^*)N_1^* + \frac{r_1^2(B_a^*)}{r_{10}} K_1'(B_a^*) \qquad (7.17c)$$

$$M_5 = \begin{bmatrix} -r_B(0, N_{2b}^*) & G_{12} & G_{13} \\ 0 & r_1(B_b^*) - \alpha N_{2b}^* & 0 \\ G_{31} & -\beta N_{2b}^* & -r_2(B_b^*) \end{bmatrix}$$

where

$$G_{12} = B_b^* \frac{\partial r_B(0, N_{2b}^*)}{\partial N_1} + \frac{r_B^2(0, N_{2b}^*)}{r_{BO}} \frac{\partial K_B(0, N_{2b}^*)}{\partial N_1} \qquad (7.18a)$$

$$G_{13} = B_b^* \frac{\partial r_B(0, N_{2b}^*)}{\partial N_2} + \frac{r_B^2(0, N_{2b}^*)}{r_{BO}} \frac{\partial K_B(0, N_{2b}^*)}{\partial N_2} \qquad (7.18b)$$

$$G_{32} = r_2'(B_b^*)N_{2b}^* + \frac{r_1^2(B^{(2)})}{r_{20}} K_2'(B_b^*) \qquad (7.18c)$$

$$M_6 = \begin{bmatrix} r_B(N_{1c}^*, N_{2c}^*) & 0 & 0 \\ H_{21} & -\frac{r_{10}N_{1c}^*}{K_{10}} & -\alpha N_1^* \\ H_{31} & -\beta N_2^* & -\frac{r_{20}N_2^*}{K_{20}} \end{bmatrix}$$

where

$$H_{21} = r_1'(0)N_{1c}^* + \frac{r_{10}N_{1c}^*}{K_{10}^2} K_1'(0) \qquad (7.19a)$$

$$H_{31} = r_2'(0)N_2^* + \frac{r_{20}N_{2c}^*}{K_{20}^2} K_2'(0) \qquad (7.19b)$$

$$M^* = \begin{bmatrix} -r_B(N_1^*,N_2^*) & I_{12} & I_{13} \\ I_{21} & -r_1(B^*) & -\alpha N_1^* \\ I_{31} & -\beta N_2^* & -r_2(B^*) \end{bmatrix}$$

where

$$I_{12} = B^* \frac{\partial r_B(N_1^*,N_2^*)}{\partial N_1} + \frac{r_B^2(N_1^*,N_2^*)}{r_{B0}} \frac{\partial K_B(N_1^*,N_2^*)}{\partial N_1} \qquad (7.20a)$$

$$I_{13} = B^* \frac{\partial r_B(N_1^*,N_2^*)}{\partial N_2} + \frac{r_B^2(N_1^*,N_2^*)}{r_{B0}} \frac{\partial K_B(N_1^*,N_2^*)}{\partial N_2} \qquad (7.20b)$$

$$I_{21} = r_1'(B^*)N_1^* + \frac{r_{10}N_1^{*2}}{K_1^2(B^*)} K_1'(B^*) \qquad (7.20c)$$

$$I_{31} = r_2'(B^*)N_2^* + \frac{r_{20}N_2^{*2}}{K_2^2(B^*)} K_2'(B^*) \qquad (7.20d)$$

From the above variational matrices we note that E_0 is locally unstable in $B-N_1-N_2$ space. E_1 is a saddle point with stable manifold locally in B direction and with unstable manifold locally in N_1-N_2 plane. E_2 is also a saddle point with stable manifold locally in N_1 direction and with unstable manifold locally in $B-N_2$ plane (Here $r_{20} - \beta K_{10}$ is taken positive). E_3 is also a saddle point with stable manifold locally in N_2 direction and with unstable manifold locally in $B-N_1$ plane (Here $r_{10} - \alpha K_{20}$ is taken positive). Again E_4 is a saddle point whose stable manifold is locally in $B-N_1$ plane and whose unstable manifold is

locally in N_2 direction. E_5 is also a saddle point whose stable manifold is locally in $B-N_2$ plane and whose unstable manifold is locally in N_1 direction. E_6 is locally unstable in B direction and locally stable in N_1-N_2 plane iff the inequality (7.11a) is satisfied.

The stability behavior of E^* is not obvious. In the following theorem we have found sufficient conditions for E^* to be locally asymptotically stable.

THEOREM 7.2.1 Let the following inequalities hold:

$$r_B(N_1^*, N_2^*) > I_{21} + I_{31} \qquad (7.21a)$$

$$r_1(B^*) > \beta N_2^* - I_{12} \qquad (7.21b)$$

$$r_2(B^*) > \alpha N_1^* - I_{13} \qquad (7.21c)$$

Then E^* is locally asymptotically stable.

Proof: If the inequalities (7.21) hold, then by Gershgorin's theorem (Lancaster and Tismanetsky(1985)) all eigen values of M^* will have negative real parts, and the theorem follows.

In the following theorem we have found certain conditions under which E^* is globally asymptotically stable. In order to prove this theorem, we first need the following lemma which establishes the region of attraction for the system (7.1). The ideas used here are developed in Hsu(1978) and Freedman(1987).

LEMMA 7.2.1 The set
$$\mathbb{R} = \left\{ (B, N_1, N_2) = 0 \leq B \leq K_{B0}, \ 0 \leq N_1 \leq N_{1d}, \ 0 \leq N_2 \leq N_{2d} \right\}$$
attracts all solutions initiating in the interior of positive orthant, where

$$N_{1d} = \frac{r_1(K_{BO})K_1(K_{BO})}{r_{10}} \quad , \quad N_{2d} = \frac{r_2(K_{BO})K_2(K_{BO})}{r_{20}} \quad .$$

Proof: From (7.1) we have

$$\frac{dB}{dt} = r_B(N_1,N_2)B - \frac{r_{BO}B^2}{K_B(N_1,N_2)}$$

$$\leq r_{BO}B - \frac{r_{BO}B^2}{K_{BO}}$$

Hence $\lim_{t \to \infty} B(t) \leq K_{BO}$

Also we have

$$\frac{dN_1}{dt} = r_1(B)N_1 - \frac{r_{10}N_1^2}{K_1(B)} - \alpha N_1 N_2$$

$$\leq r_1(K_{BO})N_1 - \frac{r_{10}N_1^2}{K_1(K_{BO})}$$

Hence $\lim_{t \to \infty} N_1(t) \leq N_{1d}$

Similarly $\lim_{t \to \infty} N_2(t) \leq N_{2d}$

<div align="center">proving the lemma</div>

THEOREM 7.2.2   In addition to the assumptions (7.2) — (7.5), let
$r_B(N_1,N_2)$, $K_B(N_1,N_2)$, $r_1(B)$, $K_1(B)$, $r_2(B)$ and $K_2(B)$ satisfy in $\mathbb{R}$

$$K_m \leq K_B(N_1,N_2) \leq K_{BO}, \quad K_{m1} \leq K_1(B) \leq K_1(K_{BO}),$$

$$K_{m2} \leq K_2(B) \leq K_2(K_{BO}), \quad 0 \leq - \frac{\partial K_B(N_1,N_2)}{\partial N_1} \leq k_1,$$

$$0 \leq - \frac{\partial K_B(N_1,N_2)}{\partial N_2} \leq k_2, \quad 0 \leq K_1'(B) \leq k_3, \quad 0 \leq K_2'(B) \leq k_4, \quad (7.22)$$

$$0 \leq - \frac{\partial r_B(N_1,N_2)}{\partial N_1} \leq \rho_1, \quad 0 \leq - \frac{\partial r_B(N_1,N_2)}{\partial N_2} \leq \rho_2,$$

$$0 \leq r_1'(B) \leq \rho_3, \quad 0 \leq r_2'(B) \leq \rho_4 \quad .$$

for some positive constants K_m, K_{m1}, K_{m2}, k_1, ρ_i, $i = 1$ to 4. Then
if the following inequalities hold

$$\left[\frac{r_{BO} K_{BO} k_1}{K_m^2} + \frac{r_{10} N_1 d^{k_3}}{K_{m1}^2} + \rho_1 + \rho_3 \right]^2 < \frac{r_{BO}}{K_B(N_1^*, N_2^*)} \frac{r_{10}}{K_1(B^*)} \qquad (7.23a)$$

$$\left[\frac{r_{BO} K_{BO} k_2}{K_m^2} + \frac{r_{20} N_2 d^{k_4}}{K_{m2}^2} + \rho_2 + \rho_4 \right]^2 < \frac{r_{BO}}{K_B(N_1^*, N_2^*)} \frac{r_{20}}{K_2(B^*)} \qquad (7.23b)$$

$$\left[\alpha + \beta \right]^2 < \frac{r_{10}}{K_1(B^*)} \frac{r_{20}}{K_2(B^*)} \qquad (7.23c)$$

E^* is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

Proof: We consider the following positive definite function about E^*,

$$V(B, N_1, N_2) = B - B^* - B^* \ln \frac{B}{B^*} + [N_1 - N_1^* - N_1^* \ln \frac{N_1}{N_1^*}]$$

$$+ [N_2 - N_2^* - N_2^* \ln \frac{N_2}{N_2^*}] \qquad (7.24)$$

Differentiating V with respect to t along the solution of (7.1), we get after some algebraic manipulations as

$$\frac{dV}{dt} = - \frac{r_{BO}}{K_B(N_1^*, N_2^*)} (B - B^*)^2 - \frac{r_{10}}{K_1(B^*)} (N_1 - N_1^*)^2 - \frac{r_{20}}{K_2(B^*)} (N_2 - N_2^*)^2$$

$$+ (B - B^*)(N_1 - N_1)[- r_{BO} B \xi_1(N_1, N_2) - r_{10} N_1 \xi_3(B)$$

$$+ \eta_1(N_1, N_2) + \eta_3(B)]$$

$$+ (B - B^*)(N_2 - N_2^*)[- r_{BO} B \xi_2(N_1^*, N_2) - r_{20} N_2 \xi_4(B)$$

$$+ \eta_2(N_1^*, N_2) + \eta_4(B)]$$

$$+ (N_1 - N_1^*)(N_2 - N_2^*)[- \alpha - \beta] \qquad (7.25)$$

where

$$\eta_1(N_1, N_2) = \begin{cases} [r_B(N_1, N_2) - r_B(N_1^*, N_2)]/(N_1 - N_1^*), & N_1 \neq N_1^* \\ \dfrac{\partial r_B(N_1^*, N_2)}{\partial N_1} & N_1 = N_1^* \end{cases} \qquad (7.26a)$$

$$\eta_2(N_1^*, N_2) = \begin{cases} [r_B(N_1^*, N_2) - r_B(N_1^*, N_2^*)]/(N_2 - N_2^*), & N_2 \neq N_2^* \\ \dfrac{\partial r_B(N_1^*, N_2^*)}{\partial N_2} & N_2 = N_2^* \end{cases} \quad (7.26b)$$

$$\eta_3(B) = \begin{cases} [r_1(B) - r_1(B^*)]/(B - B^*), & B \neq B^* \\ r_1'(B^*) & B = B^* \end{cases} \quad (7.26c)$$

$$\eta_4(B) = \begin{cases} [r_2(B) - r_2(B^*)]/(B - B^*), & B \neq B^* \\ r_2'(B^*) & B = B^* \end{cases} \quad (7.26d)$$

$$\xi_1(N_1, N_2) = \begin{cases} \left[\dfrac{1}{K_B(N_1, N_2)} - \dfrac{1}{K_B(N_1^*, N_2)}\right]/(N_1 - N_1^*), & N_1 \neq N_1^* \\ -\dfrac{1}{K_B^2(N_1^*, N_2)} \dfrac{\partial K_B(N_1^*, N_2)}{\partial N_1}, & N_1 = N_1^* \end{cases} \quad (7.26e)$$

$$\xi_2(N_1^*, N_2) = \begin{cases} \left[\dfrac{1}{K_B(N_1^*, N_2)} - \dfrac{1}{K_B(N_1^*, N_2^*)}\right]/(N_2 - N_2^*), & N_2 \neq N_2^* \\ -\dfrac{1}{K_B^2(N_1^*, N_2^*)} \dfrac{\partial K_B(N_1^*, N_2^*)}{\partial N_2}, & N_2 = N_2^* \end{cases} \quad (7.26f)$$

$$\xi_3(B) = \begin{cases} \left[\dfrac{1}{K_1(B)} - \dfrac{1}{K_1(B^*)}\right]/(B - B^*), & B \neq B^* \\ -\dfrac{K_1'(B^*)}{K_1^2(B^*)} & B = B^* \end{cases} \quad (7.26g)$$

$$\xi_4(B) = \begin{cases} \left[\dfrac{1}{K_2(B)} - \dfrac{1}{K_2(B^*)} \right] / (B - B^*), & B \neq B^* \\[3ex] - \dfrac{K_2'(B^*)}{K_2^2(B^*)} & B = B^* \end{cases} \tag{7.26h}$$

Using (7.22) and the Mean value theorem, we get

$$|\eta_1(N_1,N_2)| \leq \rho_1, \quad |\eta_2(N_1^*,N_2)| \leq \rho_2, \quad |\eta_3(B)| \leq \rho_3, \quad |\eta_4(B)| \leq \rho_4,$$

$$|\xi_1(N_1,N_2)| \leq k_1/K_m^2, \quad |\xi_2(N_1^*,N_2)| \leq k_2/K_m^2, \tag{7.27}$$

$$|\xi_3(B)| \leq k_3/K_{m1}^2, \quad |\xi_4(B)| \leq k_4/K_{m2}^2 \ .$$

Now $\dfrac{dV}{dt}$ can further be written as sum of the quadratics

$$\begin{aligned}
\frac{dV}{dt} = & - \frac{1}{2} a_{11}(B - B^*)^2 + a_{12}(B - B^*)(N_1 - N_1^*) - \frac{1}{2} a_{22}(N_1 - N_1^*)^2 \\[2ex]
& - \frac{1}{2} a_{11}(B - B^*)^2 + a_{13}(B - B^*)(N_2 - N_2^*) - \frac{1}{2} a_{33}(N_2 - N_2^*)^2 \\[2ex]
& - \frac{1}{2} a_{22}(N_1 - N_1^*)^2 + a_{23}(N_1 - N_1^*)(N_2 - N_2^*) - \frac{1}{2} a_{33}(N_2 - N_2^*)^2
\end{aligned} \tag{7.28}$$

where

$$a_{11} = \frac{r_{B0}}{K_B(N_1^*,N_2^*)} \ , \quad a_{22} = \frac{r_{10}}{K_1(B^*)} \ , \quad a_{33} = \frac{r_{20}}{K_2(B^*)} \ ,$$

$$a_{12} = - r_{B0}B \ \xi_1(N_1,N_2) - r_{10}N_1\xi_3(B) + \eta_1(N_1,N_2) + \eta_3(B)$$

$$a_{13} = - r_{B0}B \ \xi_2(N_1^*,N_2) - r_{20}N_2\xi_4(B) + \eta_2(N_1^*,N_2) + \eta_4(B)]$$

$$a_{23} = - \alpha - \beta$$

Then sufficient conditions for $\dfrac{dV}{dt}$ to be negative definite are that the following inequalities hold:

$$a_{12}^2 < a_{11} \ a_{22} \tag{7.29a}$$

$$a_{13}^2 < a_{11} \ a_{33} \tag{7.29b}$$

$$a_{23}^2 < a_{22} \ a_{33} \tag{7.29c}$$

We note that (7.23a,b,c) implies (7.29a,b,c) respectively. Hence V is a Liapunov's function with respect to E^* whose domain contains the region of attraction $\mathbb{R}$, proving the theorem.

The above theorem implies that the resource biomass will settle down to a level whose magnitude will depend upon the equilibrium level of interacting populations. It is also noted here that the increase in the density of populations will lower the density of resource biomass.

7.2.1b   THE CASE OF TWO COMPETING SPECIES WHOLLY DEPENDENT ON RESOURCE.

In this case $\alpha > 0$, $\beta > 0$, $r_i(0) = 0$, $r_i'(B) > 0$ for $B \geq 0$, $i = 1, 2$.

Here, the two competitors wholly depend upon resource. It can be checked that there are only five equilibria, namely $E_0(0,0,0)$, $E_1(K_{B0},0,0)$, $E_4(B_a^*,N_{1a}^*,0)$, $E_5(B_b^*,0,N_{2b}^*)$, $E(B^*,N_1^*,N_2^*)$. It should be noted here that the equilibria corresponding to E_2, E_3 and E_6 do not exist. This shows that in absence of the resource, no competitor can survive. It can be seen that the conditions corresponding to (7.7) and (7.9) are automatically satisfied and thus E_4 and E_5 always exist. E^* exists under the same condition(7.14). It can further be checked that the stability behavior of the equilibria is similar to the corresponding equilibria discussed in section 7.2.1a.

7.2.1c   THE CASE OF TWO COMPETING SPECIES PREDATING ON RESOURCE.

In this case $\alpha > 0$, $\beta > 0$, $r_i(0) = - r_{i0} < 0$, $r_i(B_{ci}) = 0$ for some $B_{ci} > 0$ and $r_i'(B) > 0$ for $B \geq 0$, $i = 1,2$.

Here the two populations are competing between each other but they are predating on the resource. In this case it can be checked

that there are only five equilibria, namely $E_0(0,0,0)$, $E_1(K_{BO},0,0)$, $E_4(B_a^*,N_{1a}^+,0)$, $E_5(B_b^*,0,N_{2b}^*)$ and $E(B^*,N_1^*,N_2^*)$. It should be noted here that the equilibria corresponding to E_2, E_3 and E_6 do not exist which shows that the population can not survive without resource. It can also be checked that E_4 and E_5 exist if $B_{c1} < K_{BO}$ and $B_{c2} < K_{BO}$ hold respectively. E^* exists under the same condition (7.14). The stability behavior of these equilibria is similar to the corresponding equilibria discussed in section 7.2.1a.

7.2.2a ALTERNATIVE RESOURCE DEPENDENT MODEL

In this case $\alpha = a_{12} > 0$, $\beta = -a_{21} < 0$, $r_i(0) = r_{i0} > 0$ and $r_1'(B) > 0$ for $B \geq 0$.

In this case the first population is also an alternative resource for the second population i.e. the first population may be used by the second population as a food alternative along with resource biomass. In this case the model (7.1) has also eight nonnegative equilibria as given below.

$F_0(0,0,0)$, $F_1(K_{BO},0,0)$, $F_2(0,K_{10},0)$, $F_3(0,0,K_{20})$, $F_4(\hat{B}_a,\hat{N}_{1a},0)$, $F_5(\hat{B}_b,0,\hat{N}_{1b})$, $F_6(0,\hat{N}_{1c},\hat{N}_{2c})$ and $\hat{F}(\hat{B},\hat{N}_1,\hat{N}_2)$. The equilibria F_0, F_1, F_2 and F_3 obviously exists. It can be seen that the equilibria F_4 and F_5 exist under the conditions (7.7) and (7.9) respectively. The coordinates of the equilibrium F_6 are given by

$$\hat{N}_{1c} = \frac{r_{20}K_{10}}{\delta_1}(r_{10} - a_{12}K_{20}) \qquad (7.30a)$$

$$\hat{N}_{2c} = \frac{r_{10}K_{20}}{\delta_1}(r_{20} + a_{21}K_{10}) \qquad (7.30b)$$

$$\text{where } \delta_1 = r_{10}r_{20} + a_{12}a_{21}K_{10}K_{20} \qquad (7.30c)$$

It is clear that the equilibrium F_6 exists iff

$$r_{10} > a_{12}K_{20} \qquad (7.31)$$

In the interior equilibrium $\hat{F}(\hat{B}, \hat{N}_1, \hat{N}_2)$; $\hat{B}$, $\hat{N}_1$ and $\hat{N}_2$ are the positive solution of the system of algebraic equations given below.

$$r_{B0}B = r_B(N_1, \hat{g}(B))K_B(N_1, \hat{g}(B)) \qquad (7.32a)$$

$$N_1 = \frac{K_1(B)[r_{20}r_1(B) - a_{12}r_2(B)K_2(B)]}{r_{10}r_{20} + a_{12}a_{21}K_1(B)K_2(B)} = \hat{f}(B), \quad \text{(say)} \qquad (7.32b)$$

$$N_2 = \frac{K_2(B)[r_{10}r_2(B) + a_{21}r_1(B)K_1(B)]}{r_{10}r_{20} + a_{12}a_{21}K_1(B)K_2(B)} = \hat{g}(B), \quad \text{(say)} \qquad (7.32c)$$

As earlier, it can be checked that the isoclines (7.32a) and (7.32b) will intersect at a unique point $(\hat{B}, \hat{N}_1)$, provided

$$\hat{f}'(B) > 0, \quad \hat{g}'(B) > 0 \text{ for } B \geq 0 \text{ and } \hat{N}_{1c} < \hat{N}_{10} \qquad (7.33)$$

where $\hat{N}_{1c}$ is given by (7.30a) and it is positive under the condition (7.31), and $\hat{N}_{10}$ in the interval $0 < \hat{N}_{10} < \bar{N}_1$ satisfies $r_B(\hat{N}_{10}, \hat{g}(0)) = 0$.

Thus knowing the value of $\hat{B}$ and $\hat{N}_1$, $\hat{N}_2$ can be calculated from (7.32c). It should be noted here that for for $\hat{N}_1$ to be positive, we must have

$$r_{20}r_1(\hat{B}) > a_{12}r_2(\hat{B})K_2(\hat{B}) \qquad (7.34)$$

By computing the variational matrices corresponding to each equilibrium one can check that the stability behavior of F_0 to F_5 is same as that of E_0 to E_5 respectively. F_6 is a saddle point with stable manifold locally in N_1-N_2 plane and with unstable manifold locally in B direction.

In the following theorem we state the sufficient conditions for the interior equilibrium $\hat{F}$ to be locally asymptotically stable whose proof is similar to theorem 7.2.1 and hence we omit it.

THEOREM 7.2.3 Let the following inequalities hold:

$$r_B(\hat{N}_1, \hat{N}_2) > \hat{I}_{21} + \hat{I}_{31} \qquad (7.35a)$$

$$r_1(\hat{B}) > a_{21}\hat{N}_2 - \hat{I}_{12} \qquad (7.35b)$$

$$r_2(\hat{B}) > a_{12}\hat{N}_1 - \hat{I}_{13} \qquad (7.35c)$$

Then $\hat{F}$ is locally asymptotically stable, where $\hat{I}_{1j} = I_{1j}\big|_{\hat{F}}$.

To show that $\hat{F}$ is globally asymptotically stable, we first state the following lemma which establishes the region of attraction for the system (7.1). The proof of this lemma is same as that of lemma 7.2.1 and hence we omit it.

LEMMA 7.2.2 The set

$$\hat{\mathbb{R}} = \left\{ (B, N_1, N_2) = 0 \le B \le K_{B0}, \ 0 \le N_1 \le \hat{N}_{1d}, \ 0 \le N_2 \le \hat{N}_{2d} \right\}$$

attracts all solutions initiating in the interior of positive orthant, where

$$\hat{N}_{1d} = \frac{r_1(K_{B0})K_1(K_{B0})}{r_{10}} \ , \quad \hat{N}_{2d} = \frac{[r_2(K_{B0}) + a_{21}\hat{N}_{1d}]K_2(K_{B0})}{r_{20}} \ .$$

In the following theorem we state the sufficient conditions for $\hat{F}$ to be globally stable whose proof is similar to the theorem 7.2.2 and hence we omit it.

THEOREM 7.2.4 In addition to the assumptions (7.2) — (7.5), let $r_B(N_1, N_2)$, $K_B(N_1, N_2)$, $r_1(B)$, $K_1(B)$, $r_2(B)$ and $K_2(B)$ satisfy in $\hat{\mathbb{R}}$

$$\hat{K}_m \le K_B(N_1, N_2) \le K_{B0}, \quad \hat{K}_{m1} \le K_1(B) \le K_1(K_{B0}),$$

$$\hat{K}_{m2} \le K_2(B) \le K_2(K_{B0}), \quad 0 \le -\frac{\partial K_B(N_1, N_2)}{\partial N_1} \le \hat{k}_1,$$

$$0 \le -\frac{\partial K_B(N_1, N_2)}{\partial N_2} \le \hat{k}_2, \ 0 \le K_1'(B) \le \hat{k}_3, \ 0 \le K_2'(B) \le \hat{k}_4, \qquad (7.36a)$$

$$0 \leq - \frac{\partial r_B(N_1, N_2)}{\partial N_1} \leq \hat{\rho}_1, \quad 0 \leq - \frac{\partial r_B(N_1, N_2)}{\partial N_2} \leq \hat{\rho}_2,$$

$$0 \leq r_1'(B) \leq \hat{\rho}_3, \quad 0 \leq r_2'(B) \leq \hat{\rho}_4 \tag{7.36b}$$

for some positive constants $\hat{K}_m$, $\hat{K}_{m1}$, $\hat{K}_{m2}$, $\hat{k}_1$, $\hat{\rho}_i$, $i = 1$ to 4. Then if the following inequalities hold

$$\left[\frac{r_{B0} K_{B0} \hat{k}_1}{\hat{K}_m^2} + \frac{r_{10} \hat{N}_1 d \hat{k}_3}{\hat{K}_{m1}^2} + \hat{\rho}_1 + \hat{\rho}_3 \right]^2 < \frac{r_{B0}}{K_B(\hat{N}_1, \hat{N}_2)} \frac{r_{10}}{K_1(\hat{B})} \tag{7.37a}$$

$$\left[\frac{r_{B0} K_{B0} \hat{k}_2}{\hat{K}_m^2} + \frac{a_{12} r_{20} \hat{N}_2 d \hat{k}_4}{a_{21} \hat{K}_{m2}^2} + \hat{\rho}_2 + \frac{a_{12} \hat{\rho}_4}{a_{21}} \right]^2 < \frac{a_{12} r_{B0}}{a_{21} K_B(\hat{N}_1, \hat{N}_2)} \frac{r_{20}}{K_2(\hat{B})} \tag{7.37b}$$

$\hat{F}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

REMARK: In the proof of the above theorem, the following Liapunov's function is considered:

$$W(B, N_1, N_2) = [B - \hat{B} - \hat{B} \ln \frac{B}{\hat{B}}] + [N_1 - \hat{N}_1 - \hat{N}_1 \ln \frac{N_1}{\hat{N}_1}]$$

$$+ \frac{a_{12}}{a_{21}} [N_2 - \hat{N}_2 - \hat{N}_2 \ln \frac{N_2}{\hat{N}_2}] \tag{7.38}$$

The above theorem also shows that the density of the resource biomass depends upon the steady state of interacting populations and the effects of populations are to decrease the density of biomass.

7.2.2b   THE TWO POPULATIONS WHOLLY DEPEND ON RESOURCE BUT ONE POPULATION ACTS AS A RESOURCE FOR THE OTHER.

In this case $\alpha = a_{12} > 0$, $\beta = - a_{21} < 0$, $r_i(0) = 0$ and $r_i'(B) > 0$ for $B \geq 0$, $i = 1, 2$.

Here, the two populations wholly depend upon the resource and the population of density $N_2(t)$ also depends upon the population

of density $N_1(t)$. In this case, there will be only five
equilibria, namely $F_0(0,0,0)$, $F_1(K_{B0},0,0)$, $F_4(\hat{B}_a,\hat{N}_{1a},0)$,
$F_5(\hat{B}_b,0,\hat{N}_{2b})$ and $\hat{F}(\hat{B},\hat{N}_1,\hat{N}_2)$. Here also the equilibria
corresponding to F_2, F_3 and F_6 do not exist. Since N_1 depends
wholly on B and N_2 depends on B as well as on N_1 and thus non
existence of the equilibria corresponding to F_2, F_3 and F_6 shows
that no population can survive in the absence of the resource. It
can also be checked that the conditions corresponding to (7.7) and
(7.9) are satisfied and hence F_4 and F_5 always exist. $\hat{F}$ exists.
under the same condition (7.33). The stability behavior of the
equilibria is similar to the corresponding equilibria discussed in
section 7.2.2a.

7.2.2c THE TWO POPULATIONS PREDATING ON RESOURCE AND ONE POPULATION IS AN ALTERNATIVE RESOURCE FOR THE OTHER.

In this case $\alpha = a_{12} > 0$, $\beta = -a_{21} < 0$, $r_1(0) = -r_{i0} < 0$,
$r_1(B_{c1}) = 0$ for some $B_{c1} > 0$ and $r_i'(B) > 0$ for $B \geq 0$, $i = 1,2$.

It can be checked that there are only five equilibria, namely
$F_0(0,0,0)$, $F_1(K_{B0},0,0)$, $F_4(\hat{B}_a,\hat{N}_{1a},0)$, $F_5(\hat{B}_b,0,\hat{N}_{2b})$ and $\hat{F}(\hat{B},\hat{N}_1,\hat{N}_2)$.
The equilibria corresponding to F_2, F_3 and F_6 do not exist. F_4 and
F_5 exist if $B_{c1} < K_{B0}$ and $B_{c2} < K_{B0}$ hold respectively. $\hat{F}$ exists
under the same condition (7.33). The stability behavior of these
equilibria is similar to the corresponding equilibria discussed in
section 7.2.2a.

7.2.3a THE TWO POPULATIONS ACT AS PREDATOR-PREY WITH ALTERNATIVE RESOURCE.

In this case $\alpha = a_{12} > 0$, $\beta = -a_{21} < 0$, $r_1(0) = r_{10} > 0$,
$r_1'(B) > 0$, $r_2(0) = -r_{20} < 0$, $r_2(B_{c2}) = 0$ for some $B_{c2} > 0$ and
$r_2'(B) > 0$ for $B \geq 0$.

In this case the model (7.1) has seven nonnegative equilibria as given below.

$\bar{F}_0(0,0,0)$, $\bar{F}_1(K_{B0},0,0)$, $\bar{F}_2(0,K_{10},0)$, $\bar{F}_3(\bar{B}_a,\bar{N}_{1a},0)$, $\bar{F}_4(\bar{B}_b,0,\bar{N}_{1b})$, $\bar{F}_5(0,\bar{N}_{1c},\bar{N}_{2c})$ and $\bar{F}(\bar{B}_d,\bar{N}_{1d},\bar{N}_{2d})$. It is clear that $\bar{F}_0$, $\bar{F}_1$ and $\bar{F}_2$ always exist. It can be seen that $\bar{F}_3$ exists under the condition (7.7) and $\bar{F}_4$ exists, provided

$$B_{c2} < K_{B0} \tag{7.39}$$

The coordinates of $\bar{F}_5$ are given by

$$\bar{N}_{1c} = \frac{r_{20}K_{10}}{\delta_1}(r_{10} + a_{12}K_{20}) \tag{7.40a}$$

$$\bar{N}_{2c} = \frac{r_{10}K_{20}}{\delta_1}(a_{21}K_{10} - r_{20}) \tag{7.40b}$$

where $\delta_1 = r_{10}r_{20} + a_{12}a_{21}K_{10}K_{20}$ $\qquad(7.40c)$

Clearly $\bar{N}_{2c} > 0$ iff

$$K_{10} > r_{20}/a_{21} \tag{7.41}$$

The coordinates of $\bar{F}$ are given by the coordinates of $\hat{F}$ in case II.

By computing the variational matrices corresponding to each equilibrium, it can be seen that $\bar{F}_0$ is a saddle point with stable manifold locally in N_2 direction and with unstable manifold locally in B-N_1 plane. Further the local stability behaviors of $\bar{F}_1$, $\bar{F}_2$, $\bar{F}_3$, $\bar{F}_4$ and $\bar{F}_5$ are same as that of F_1, F_2, F_4, F_5 and F_6 respectively as in section 7.2.2a. The local and global stability behavior of $\bar{F}$ is same as that of $\hat{F}$ in section 7.2.2a.

7.2.3b THE TWO POPULATIONS PREDATING ON RESOURCE AND ACTING AS PREDATOR-PREY BETWEEN EACH OTHER.

In this case $\alpha = a_{12} > 0$, $\beta = -a_{21} < 0$, $r_i(0) = -r_{i0} < 0$, $r_i(B_{ci}) = 0$ for some $B_{c1} > 0$ and $r_i'(B) > 0$ for $B \geq 0$, $i = 1,2$.

It can be checked that there are only five equilibria, namely

$\bar{F}_0(0,0,0)$, $\bar{F}_1(K_{BO},0,0)$, $\bar{F}_3(\bar{B}_a,\bar{N}_{1a},0)$, $\bar{F}_4(\bar{B}_b,0,\bar{N}_{1b})$ and $\bar{F}(\bar{B}_d,\bar{N}_{1d},\bar{N}_{2d})$. The equilibria corresponding to $\bar{F}_2$ and $\bar{F}_5$ do not exist. $\bar{F}_3$ and $\bar{F}_4$ exist if $B_{c1} < K_{BO}$ and $B_{c2} < K_{BO}$ hold respectively. $\bar{F}$ exists under the same condition as that of $\hat{F}$ in section 7.2.2a. The stability behavior of these equilibria is similar to the corresponding equilibria discussed in section 7.2.3a.

7.2.4a   THE CASE OF TWO COOPERATING SPECIES WITH ALTERNATIVE RESOURCE.

In this case $\alpha = -b_{12} < 0$, $\beta = -b_{21} < 0$, $r_i(0) = r_{i0} > 0$ and $r_i'(B) > 0$ for $B \geq 0$, $i = 1,2$.

In this case the model (7.1) has also eight equilibria as given below.

$G_0(0,0,0)$, $G_1(K_{BO},0,0)$, $G_2(0,K_{10},0)$, $G_3(0,0,K_{20})$, $G_4(\tilde{B}_a,\tilde{N}_{1a},0)$, $G_5(\tilde{B}_b,0,\tilde{N}_{2b})$, $G_6(0,\tilde{N}_{1c},\tilde{N}_{2c})$ and $\tilde{G}(\tilde{B},\tilde{N}_1,\tilde{N}_2)$. The equilibria G_0, G_1, G_2 and G_3 obviously exist. The equilibria G_4 and G_5 exist under the same conditions (7.7) and (7.9) respectively. It can be checked that the coordinates of the equilibrium G_6 are given by

$$\tilde{N}_{1c} = \frac{r_{20}K_{10}}{\delta_1} (r_{10} + b_{12}K_{20}) \tag{7.42a}$$

$$\tilde{N}_{2c} = \frac{r_{10}K_{20}}{\delta_1} (r_{20} + b_{21}K_{10}) \tag{7.42b}$$

where $\delta_1 = r_{10}r_{20} - b_{12}b_{21}K_{10}K_{20}$ \hfill (7.42c)

and the equilibrium G_6 exists iff

$$\delta_1 > 0 \tag{7.43}$$

Finally, in the interior equilibrium $\tilde{G}(\tilde{B},\tilde{N}_1,\tilde{N}_2)$, $\tilde{B}$, $\tilde{N}_1$ and $\tilde{N}_2$ are the positive solution of the system of algebraic equations as given below.

$$r_{B0}B = r_B(N_1, \tilde{g}(B))K_B(N_1, \tilde{g}(B)) \tag{7.44a}$$

$$N_1 = \frac{K_1(B)[r_{20}r_1(B) + b_{12}r_2(B)K_2(B)]}{r_{10}r_{20} - b_{12}b_{21}K_1(B)K_2(B)} = \tilde{f}(B), \quad \text{(say)} \tag{7.44b}$$

$$N_2 = \frac{K_2(B)[r_{10}r_2(B) + b_{21}r_1(B)K_1(B)]}{r_{10}r_{20} - b_{12}b_{21}K_1(B)K_2(B)} = \tilde{g}(B), \quad \text{(say)} \tag{7.44c}$$

It should be noted from (7.44) that as the interspecific coefficients b_{12} and b_{21} increase, N_1 and N_2 also increase and hence B decreases. We also note from (7.42) and (7.44) that the densities of interacting populations in (7.44) are greater than the densities of interacting populations in (7.42) i.e $\tilde{N}_1 > \tilde{N}_{1c}$ and $\tilde{N}_2 > \tilde{N}_{2c}$.

As before, it can also be checked that the isoclines (7.44a) and (7.44b) will intersect at a unique point $(\tilde{B}, \tilde{N}_1)$, provided

$$\tilde{f}'(B) > 0, \ \tilde{g}'(B) > 0 \text{ for } B \geq 0 \text{ and } \tilde{N}_{1c} < \tilde{N}_{10} \tag{7.45}$$

where $\tilde{N}_{1c}$ is given by (7.42a) and it is positive under the condition (7.43) and $\tilde{N}_{10}$ in the interval $0 < \tilde{N}_{10} < \bar{N}_1$ satisfies $r_B(\tilde{N}_{10} \ \tilde{g}(0)) = 0$.

Thus knowing the value of $\tilde{B}$ and $\tilde{N}_1$, $\tilde{N}_2$ can be calculated from (7.44c). It should be noted here that for for $\tilde{N}_1$ and $\tilde{N}_2$ to be positive, we must have

$$r_{10}r_{20} > b_{12}b_{21}K_1(\tilde{B})K_2(\tilde{B}) \tag{7.46}$$

Again by computing the variational matrices corresponding to each equilibrium, one can check that the stability behavior of G_0 to G_6 coincides with the stability behavior of E_0 to E_6 respectively.

In the following theorem we state the sufficient conditions for $\tilde{G}$ to be locally asymptotically stable whose proof is also

similar to theorem 7.2.1 and hence is omitted.

THEOREM 7.2.5 Let the following inequalities hold:

$$r_B(\tilde{N}_1, \tilde{N}_2) > \tilde{I}_{21} + \tilde{I}_{31} \tag{7.47a}$$

$$r_1(\tilde{B}) > b_{21}\tilde{N}_2 - \tilde{I}_{12} \tag{7.47b}$$

$$r_2(\tilde{B}) > b_{12}\tilde{N}_1 - \tilde{I}_{13} \tag{7.47c}$$

Then $\tilde{G}$ is locally asymptotically stable, where $\tilde{I}_{ij} = I_{ij}\big|_{\tilde{G}}$.

The following lemma gives the criteria for the region of attraction for the system (7.1).

LEMMA 7.2.3 The set
$$\tilde{R} = \left\{ (B, N_1, N_2): 0 \le B \le K_{BO}, \ 0 \le N_1 < \infty, \ 0 \le N < \infty \right\} \text{ attracts}$$

all solution initiating in the interior of the positive orthant,

Proof: As before, $\lim_{t \to \infty} B(t) \le K_{BO}$

We also have

$$\dot{N}_1 = r_1(B)N_1 - \frac{r_{10}N_1^2}{K_1(B)} + b_{12}N_1N_2$$

$$\le r_1(K_{BO})N_1 - \frac{r_{10}N_1^2}{K_1(K_{BO})} + b_{12}N_1N_2$$

Similarly,

$$\dot{N}_2 \le r_2(K_{BO})N_2 - \frac{r_{20}N_2^2}{K_2(K_{BO})} + b_{21}N_1N_2$$

Let u be a positive real number.

Then we have

$$\frac{\dot{N}_1}{N_1} + u\frac{\dot{N}_2}{N_2} \le (a_1 + ua_2) - (b_1 - ub_{21})N_1 - (ub_2 - b_{12})N_2$$

where $a_1 = r_1(K_{BO})$, $b_i = r_{i0}/K_i(K_{BO})$, $i = 1, 2$.

Choose $u > 0$ such that

$$b_{12}/b_2 < u < b_1/b_{21}$$

Then $\ln \left(\dfrac{N_1 N_2^u}{N_{10} N_{20}^u} \right) < (a_1 + u a_2)t$

Hence $\lim\limits_{t \to \infty} N_1 N_2^u < \infty$.

proving the lemma.

In the following theorem we state the sufficient conditions under which $\tilde{G}$ is globally stable whose proof is similar to theorem 7.2.2 and hence is omitted.

THEOREM 7.2.6   In addition to the assumptions (7.2) — (7.5), let $r_B(N_1, N_2)$, $K_B(N_1, N_2)$, $r_1(B)$, $K_1(B)$, $r_2(B)$ and $K_2(B)$ satisfy in $\tilde{\mathbb{R}}$

$$\tilde{K}_m \leq K_B(N_1, N_2) \leq K_{B0}, \quad \tilde{K}_{m1} \leq K_1(B) \leq K_1(K_{B0}),$$

$$\tilde{K}_{m2} \leq K_2(B) \leq K_2(K_{B0}), \quad 0 \leq -\frac{\partial K_B(N_1, N_2)}{\partial N_1} \leq \tilde{k}_1,$$

$$0 \leq -\frac{\partial K_B(N_1, N_2)}{\partial N_2} \leq \tilde{k}_2, \quad 0 \leq K_1'(B) \leq \tilde{k}_3, \quad 0 \leq K_2'(B) \leq \tilde{k}_4, \quad (7.48)$$

$$0 \leq -\frac{\partial r_B(N_1, N_2)}{\partial N_1} \leq \tilde{\rho}_1, \quad 0 \leq -\frac{\partial r_B(N_1, N_2)}{\partial N_2} \leq \tilde{\rho}_2,$$

$$0 \leq r_1'(B) \leq \tilde{\rho}_3, \quad 0 \leq r_2'(B) \leq \tilde{\rho}_4$$

for some positive constants $\tilde{K}_m$, $\tilde{K}_{m1}$, $\tilde{K}_{m2}$, $\tilde{k}_i$, $\tilde{\rho}_i$, $i = 1$ to 4. Then if the following inequalities hold

$$\left[\frac{r_{B0} K_{B0} \tilde{k}_1}{\tilde{K}_m^2} + \frac{r_{10} \tilde{N}_1 \tilde{k}_3}{\tilde{K}_{m1}^2} + \tilde{\rho}_1 + \tilde{\rho}_3 \right]^2 < \frac{r_{B0}}{K_B(\tilde{N}_1, \tilde{N}_2)} \frac{r_{10}}{K_1(K_{B0})} \quad (7.49a)$$

$$\left[\frac{r_{B0} K_{B0} \tilde{k}_2}{\tilde{K}_m^2} + \frac{r_{20} \tilde{N}_2 \tilde{k}_4}{\tilde{K}_{m2}^2} + \tilde{\rho}_2 + \tilde{\rho}_4 \right]^2 < \frac{r_{B0}}{K_B(\tilde{N}_1, \tilde{N}_2)} \frac{r_{20}}{K_2(K_{B0})} \quad (7.49b)$$

$$\left[b_{12} + b_{21} \right]^2 < \frac{r_{10}}{K_1(K_{B0})} \frac{r_{20}}{K_2(K_{B0})} \qquad (7.49c)$$

$\hat{F}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.

This theorem implies that if the inequalities (7.49) hold, the resource biomass will settle down to a level the magnitude of which will depend upon the equilibrium level of two cooperative populations. It is also noted here that in the case of cooperation, the depletion of resource biomass is more than the three cases discussed earlier. It should be also pointed out here that in this case the density of interacting populations is more than the case when the interspecific coefficients b_{12} and b_{21} are zero. This shows that the introduction of interspecific coefficients cause more depletion of resource biomass. It should be also noted here that if the pressure due to populations on the resource biomass increases unabatedly the biomass may be threatened to extinction.

7.2.4b THE CASE OF TWO COOPERATING SPECIES WHOLLY DEPENDENT ON RESOURCE.

In this case $\alpha = - b_{12} < 0$, $\beta = - b_{21} < 0$, $r_i(0) = 0$ and $r_i'(B) > 0$ for $B \geq 0$, $i = 1,2$.

Here, the two cooperating populations wholly depend upon the resource. It can be checked that there are only five equilibria, namely $G_0(0,0,0)$, $G_1(K_{B0},0,0)$, $G_4(\tilde{B}_a,\tilde{N}_{1a},0)$, $G_5(\tilde{B}_b,0,\tilde{N}_{2b})$ and $\tilde{G}(\tilde{B},\tilde{N}_1,\tilde{N}_2)$. Here also the equilibria corresponding to G_2, G_3 and G_6 do not exist. This shows that even in the case of cooperation no population will survive without resource. It can be checked that the equilibria G_4 and G_5 always exist. $\tilde{G}$ exists under the

same condition (7.45). The stability behavior of the equilibria is similar to the corresponding equilibria discussed in section 7.2.4a.

7.2.4c THE CASE OF TWO COOPERATING SPECIES PREDATING ON RESOURCE.

In this case $\alpha = -b_{12} < 0$, $\beta = -b_{21} < 0$, $r_i(0) = -r_{i0} < 0$, $r_1(B_{c1}) = 0$ for some $B_{c1} > 0$ and $r_1'(B) > 0$ for $B \geq 0$, $i = 1,2$.

It can be checked that there are only six equilibria, namely $G_0(0,0,0)$, $G_1(K_{B0},0,0)$, $G_4(\tilde{B}_a,\tilde{N}_{1a},0)$, $G_5(\tilde{B}_b,0,\tilde{N}_{2b})$, $G_6(0,\tilde{N}_{1c},\tilde{N}_{2c})$ and $\tilde{G}(\tilde{B},\tilde{N}_1,\tilde{N}_2)$. The equilibria corresponding to G_2 and G_3 do not exist. G_4 and G_5 exist if $B_{c1} < K_{B0}$ and $B_{c2} < K_{B0}$ hold respectively. G_6 exists if and only of the following holds.

$$b_{12}b_{21}K_{10}K_{20} > r_{10}r_{20}$$

$\tilde{G}$ exists under the same condition (7.45). The stability behavior of these equilibria is similar to the corresponding equilibria discussed in section 7.2.4a.

7.3 EXAMPLES

In this section we give an example in each case by selecting the functions given below.

$$r_B(N_1,N_2) = r_{B0} - r_{B1}N_1 - r_{B2}N_2$$
$$K_B(N_1,N_2) = K_{B0} - K_{B1}N_1 - K_{B2}N_2$$
$$r_1(B) = r_{10} + r_{11}B$$
$$r_2(B) = r_{20} + r_{21}B \tag{7.50}$$
$$K_1(B) = K_{10} + K_{11}B$$
$$K_2(B) = K_{20} + K_{21}B$$

In each case we choose the following set of parameters:

$$r_{B1} = 0.04, \ r_{B2} = 0.05, \ K_{B0} = 4.9109, \ K_{B1} = 0.01, \ K_{B2} = 0.02,$$
$$r_{10} = 10, \ r_{11} = 0.03, \ r_{20} = 20, \ r_{21} = 0.05, \ K_{10} = 5, \ K_{11} = 0.03,$$
$$K_{20} = 8, \ K_{21} = 0.04. \tag{7.51}$$

CASE I: The case of competition

Here by choosing $\alpha = 0.2$, $\beta = 0.3$, $r_{BO} = 12.5$, it can be checked that the condition (7.14) for the existence of $E^*(B^*, N_1^*, N_2^*)$ is satisfied and thus E^* exists and is given by

$$B^* \simeq 4.5, \quad N_1^* \simeq 4.4104, \quad N_2^* \simeq 7.7309$$

It can also be checked that the conditions (7.21) are satisfied which shows that E^* is locally asymptotically stable.

Choosing $K_m = 2 = K_{m1} = K_{m2}$ in theorem 7.2.2, it can be seen that the conditions (7.23) are satisfied and hence E^* is globally asymptotically stable in the region

$$\mathbb{R} = \left\{ (B, N_1, N_2) = 0 \leq B \leq 4.9109, \ 0 \leq N_2 \leq 5.4671, \ 0 \leq N_2 \leq 8.6208 \right\}$$

CASE II: The case of another alternative resource

In this case we choose the following parameters:

$$\alpha = a_{12} = 0.2, \quad \beta = -a_{21} = -0.3, \quad r_{BO} = 4.8312.$$

Under the above set of parameters, it can be checked that $\hat{F}$ exists and its coordinates are given by

$$\hat{B} \simeq 4.1, \quad \hat{N}_1 \simeq 4.2872, \quad \hat{N}_2 \simeq 8.7727 \ .$$

It can also be seen that the inequalities (7.35) in theorem 7.2.3 are satisfied and hence $\hat{F}$ is locally asymptotically stable.

By selecting $\hat{K}_m = 2 = \hat{K}_{m1}$, $\hat{K}_{m2} = 2.5$ in theorem 7.2.4, it can be seen that the inequalities (7.37) are satisfied and hence $\hat{F}$ is globally asymptotically stable in the region

$$\hat{\mathbb{R}} = \left\{ (B, N_1, N_2) = 0 \leq B \leq 4.9109, \ 0 \leq N_2 \leq 5.2231, \ 0 \leq N_2 \leq 8.9392 \right\}$$

CASE III: The case of predation

In the model (7.1), we take

$$r_2(B) = -0.1 + 0.5 B, \quad r_{20} = 2, \quad \alpha = 0.2, \quad \beta = -0.3, \quad r_{BO} = 12.0147.$$

We choose the values of the other set of parameters in the model (7.1) as same as in (7.50) and (7.51). Then it can be checked that the interior equilibrium $\bar{F}(\bar{B}, \bar{N}_{1d}, \bar{N}_{2d})$ exists and is given by

$$\bar{B} \simeq 4.3, \quad \bar{N}_{1d} \simeq 3.8516, \quad \bar{N}_{2d} \simeq 13.0976 \ .$$

It can also be checked that the conditions corresponding to (7.35) in theorem 7.2.3 are satisfied showing that $\bar{F}$ is locally asymptotically stable.

In the region

$$\mathbb{R} = \{(B, N_1, N_2): 0 \leq B \leq 4.9109, \ 0 \leq N_1 \leq 5.2231, \ 0 \leq N_2 \leq 16.0749\}$$

by assuming

$$2.5 \leq K_B(N_1, N_2) \leq 4.9109 = K_{BO}, \quad 1 \leq K_1(B) \leq 5.1437 = K_1(K_{BO}),$$

$$3.5 \leq K_2(B) \leq 8.1964 = K_2(K_{BO})$$

it can further be checked that $\hat{F}$ is globally asymptotically stable in the above region.

CASE IV: The case of cooperation

Select $\alpha = - b_{12} = - 0.2$, $\beta = - b_{21} = - 0.3$, $r_{BO} = 2.7542$. Then one can check that $\widetilde{G}(\widetilde{B}, \widetilde{N}_1, \widetilde{N}_2)$ exists and is given by

$$\widetilde{B} \simeq 3.5, \quad \widetilde{N}_1 \simeq 6.0727, \quad \widetilde{N}_2 = 8.9527.$$

It can be seen that $\widetilde{G}$ is locally asymptotically stable.
Choosing $\widetilde{K}_m = 2 = \widetilde{K}_{m1}$, $\widetilde{K}_{m2} = 3$ in theorem 7.2.6, one can see that $\widetilde{G}$ is globally asymptotically stable.

7.4   CONSERVATION MODEL

In this section, a model to conserve the density of resource and to control the densities of populations is proposed. Let $F(t)$ be the density of effort applied to conserve the density of resource and $F_1(t)$, $F_2(t)$ be the densities of efforts applied to control the populations of densities $N_1(t)$, $N_2(t)$ respectively. It is assumed that $F(t)$ is proportional to the depleted level of

resource density and $F_1(t)$, $F_2(t)$ are proportional to the undesired level of populations of densities $N_1(t)$, $N_2(t)$ respectively. Then the dynamics of the system can be written as

$$\frac{dB}{dt} = r_B(N_1,N_2)B - \frac{r_{B0}B^2}{K_B(N_1,N_2)} + r_0F$$

$$\frac{dN_1}{dt} = r_1(B)N_1 - \frac{r_{10}N_1^2}{K_1(B)} - \alpha N_1N_2 - r_{11}N_1F_1$$

$$\frac{dN_2}{dt} = r_2(B)N_2 - \frac{r_{20}N_2^2}{K_2(B)} - \beta N_1N_2 - r_{22}N_2F_2 \qquad (7.52)$$

$$\frac{dF}{dt} = r\left(1 - \frac{B}{K_{B0}}\right) - \nu F$$

$$\frac{dF_1}{dt} = r_1(N_1 - N_{1c}) - \nu_1F_1$$

$$\frac{dF_2}{dt} = r_2(N_2 - N_{2c}) - \nu_2F_2$$

$B(0) \geq 0$, $N_1(0) \geq 0$, $N_2(0) \geq 0$, $F(0) \geq 0$, $F_1(0) \geq 0$, $F_2(0) \geq 0$.

Here r, r_1, r_2 are the growth rate coefficients of the efforts $F(t)$, $F_1(t)$, $F_2(t)$ respectively and ν, ν_1, ν_2 are their respective depletion rate coefficients. r_0 is the growth rate coefficient of resource biomass, r_{11} and r_{22} are the depletion rate coefficients of $N_1(t)$ and $N_2(t)$ respectively. Further, N_{1c} and N_{2c} are the critical value of $N_1(t)$ and $N_2(t)$ respectively which are harmless to the resource density. Other notations in the model (7.52) have the same meaning as in the model (7.1).

The model (7.52) can be analysed as in the previous sections in different cases depending upon the nature of the interaction of the populations. In particular, the above model (7.52) shows that if efforts to conserve the density of resource and to control the densities of populations are suitably applied, then an appropriate level of resource density can be maintained.

7.5 CONCLUSIONS

In this chapter, a mathematical model to study the effects of two interacting populations on depletion of resource biomass is proposed and analysed. The two cases i.e. when the resource is an alternative resource for the populations and when populations wholly depend upon the resource are presented. It is assumed that the dynamics of resource biomass and populations are governed by generalized logistic equations. It is further assumed that the growth rate and carrying capacity of resource biomass decrease with increase in the density of populations. It is also assumed that the growth rate and carrying capacity of populations increase as the density of resource biomass increases. It is considered that two populations interact each other in four different ways, namely (i) competition, (ii) alternative resource, (iii) predation and (iv) cooperation. It is further considered that the two interacting population utilize the resource in three different ways, namely (i) the two interacting species with an alternative resource, (ii) the two interacting species being wholly dependent on resource and (iii) the two interacting species predating on resource. In each case local and global stability character of the system is studied. In each case it is noted that it is worthwhile to incorporate the intraspecific coefficients in a resource based ecological model with interacting populations. By analysing the model it is shown that in each case the resource biomass settles down to a level whose magnitude depends upon the equilibrium level of interacting populations. It is also noted that in the case of competition the density of resource biomass will decrease slowly. In the case when first population also behaves as another

alternative resource for the second population, then the level of the resource biomass will be lower than the competition case and in the case of cooperation the level of resource biomass will be lowest as compared to other two previous cases, other parameters and functions in the system being same. It is noted here that the resource biomass may not last long if the pressure due to populations increases without control. It is also noted that equilibrium level of the resource density in the case of cooperation is minimum and in the case of competition is maximum. Further a conservation model is also proposed to show that by conserving the resource density and by controlling population densities, an appropriate level of resource biomass can be maintained.
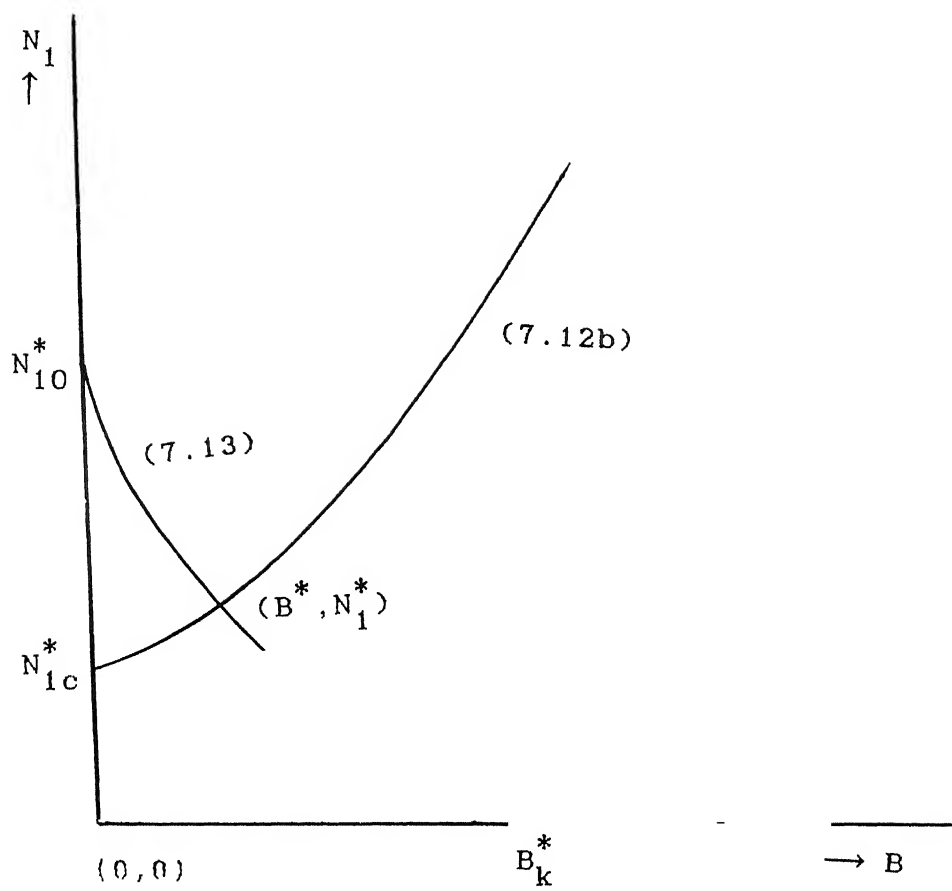
Fig. 7.1

CHAPTER VIII

MODELLING THE DEPLETION AND CONSERVATION OF FORESTRY RESOURCES: EFFECT OF POPULATION ON SURVIVAL OF WILDLIFE SPECIES

8.0 INTRODUCTION

Due to rapid industrialization and associated increase of population in forested lands and valleys, the depletion of forest resource biomass caused by expansion of agricultural land, cutting of trees in the forest for industrial purposes, use of biomass for fuel, fodder and paper industries has lead to the extinction or migration of number of species affecting the biodiversity of the habitat. There are many such ecologically susceptible regions around the world and one such region is Doon Valley in the northern province of Uttar Pradesh in India. As pointed out earlier, the main reasons for the depletion of forest biomass and threat to ecological stability in the Doon Valley are the increase in human and cattle populations and associated industrial development, Munn and Fedorov (1986), Shukla et al. (1989). This causes not only a decrease in the growth rate density of resource dependent wildlife species but also the extinction of some species and migration of others to the upper reaches of Himalayas.

Keeping in view of the above in this chapter an attempt is made to model the effect of such changes in the habitat on growth and survival of wildlife species, that are caused by depletion of forest resource in the habitat due to pressures of both human and cattle populations and decrease in the carrying capacity of the habitat, Munn and Fedorov (1986). We use here the stability theory to analyse the model for predicting the survival of biomass dependent species in the habitat.

8.1 MATHEMATICAL MODEL

Consider the forested habitat as a simple closed region where we wish to model the survival of wildlife species dependent on forest resources which are depleted due to rising populations pressures. We assume that the cumulative density of wildlife species is dependent upon the biomass density in the forested habitat and its growth rate increases as the density of the forest biomass increases. We consider that the carrying capacity of the habitat with respect to the wildlife species is dependent on population density and decreases with it. It is assumed further that the growth rate of density of the forest biomass decreases with the density of wildlife population as well as with the cumulative density of human and cattle populations. The growth rate of human and cattle populations increases as the density of biomass increases.

In view of the above and assuming that the density of wildlife species is governed by the generalized logistic type equation, the mathematical model governing the system can be written as

$$\frac{dN}{dt} = r(B)N - \frac{r_0 N^2}{K(P)}$$

$$\frac{dB}{dt} = B \, g(B) - \beta \, N \, p(B) - \alpha \, P \, B \qquad\qquad (8.1)$$

$$\frac{dP}{dt} = r_P(1 - \frac{P}{L})P + \alpha_1 \, P \, B$$

$$N(0) = N_0 \geq 0, \quad B(0) = B_0 \geq 0, \quad P(0) = P_0 \geq 0.$$

where $N(t)$ is the cumulative density of wildlife species, $B(t)$ is the density of the biomass and $P(t)$ is the cumulative density of human and cattle populations.

Here, r(B) is the growth rate coefficient of wildlife species and increases as biomass density increases and it satisfies the following conditions:

Case I: $r(0) = r_0 > 0$, $r'(B) \geq 0$ for $B \geq 0$ (8.2a)

In this case the biomass is an alternative resource for the wildlife species.

Case II: $r(0) = 0$, $r'(B) \geq 0$ for $B \geq 0$ (8.2b)

In this case the wildlife species wholly depends upon the biomass.

Case III: $r(0) = - r_0 < 0$, $r(B_a) = 0$ for some $B_a > 0$, $r'(B) \geq 0$ for $B \geq 0$ (8.2c)

In this case the wildlife species acts as a predator on the resource

The function K(P) is the maximum density of wildlife population which the environment can support and it decreases as the density of human and cattle populations increases i.e.

$$K(0) = K_0 > 0,   K'(P) \leq 0 \text{ for } P \geq 0 \qquad (8.3)$$

The function g(B) is the specific growth rate of biomass which decreases as B increases and hence

$$g(0) = g_0 > 0 ,   g'(B) < 0$$
$$\text{and}   g(K_B) = 0   \text{for some } K_B > 0 \qquad (8.4)$$

The function p(B) is functional response type function and it satisfies

$$p(0) = 0 ,   p'(B) > 0   \text{for } B \geq 0 \qquad (8.5)$$

Also r_P and L are the growth rate coefficient and carrying capacity respectively for human and cattle population density. All these constant are assumed to be positive.

We note from (8.1) that in absence of interactions, each of

these equations will reduce to logistic form.

8.2 MATHEMATICAL ANALYSIS

CASE I: BIOMASS BEING AN ALTERNATIVE RESOURCE FOR THE WILDLIFE SPECIES

From (8.1) the equilibria are given by the following equations,

$$r(B)N - \frac{r_0 N^2}{K(P)} = 0$$

$$B\, g(B) - \beta N\, p(B) - \alpha P B = 0 \tag{8.6}$$

$$r_p(1 - \frac{P}{L})P + \alpha_1 P B = 0$$

On solving these equations we get eight equilibria i.e.

$E_0(0,0,0)$, $E_1(K_0,0,0)$, $E_2(0,K_B,0)$, $E_3(0,0,L)$, $E_4(K(L),0,L)$, $E_5(\bar{N},\bar{B},0)$, $E_6(0,\tilde{B},\tilde{P})$, and $E^*(N^*,B^*,P^*)$.

Existence of $E_5(\bar{N},\bar{B},0)$: Here $\bar{N}$ and $\bar{B}$ are given by

$$\bar{N} = \frac{r(\bar{B})\, K_0}{r_0} \tag{8.7a}$$

$$\bar{N} = \frac{\bar{B}\, g(\bar{B})}{\beta\, p(\bar{B})} \tag{8.7b}$$

From (8.7a) we note that
$$\lim_{\bar{B}\to 0} \bar{N} = K_0 \tag{8.8a}$$

$$\frac{d\bar{N}}{d\bar{B}} = \frac{r'(\bar{B})\, K_0}{r_0} > 0 \tag{8.8b}$$

From (8.7b) we note that
$$\lim_{\bar{B}\to 0} \bar{N} = \frac{g_0}{\beta\, p'(0)} \tag{8.9a}$$

$$\lim_{\bar{B}\to K_B} \bar{N} = 0 \tag{8.9b}$$

$$\frac{d\bar{N}}{d\bar{B}} = \frac{1}{\beta\, p^2(\bar{B})} [\; p(\bar{B})\; g(\bar{B}) + \bar{B}\{p(\bar{B})\; g'(\bar{B}) - p'(\bar{B})\; g(\bar{B})\}] \qquad (8.9c)$$

Thus from (8.7a) and (8.8a,b) we note that $\bar{N}$ is an increasing function of $\bar{B}$ starting from K_0.

From (8.9c) we note that $\dfrac{d\bar{N}}{d\bar{B}}$ is negative provided

$$\frac{1}{\bar{B}} + \frac{g'(\bar{B})}{g(\bar{B})} \;<\; \frac{p'(\bar{B})}{p(\bar{B})}$$

From (8.7b) and (8.9a,b,c) we now see that $\bar{N}$ is a decreasing function of $\bar{B}$ starting from $g_0/\beta\, p'(0)$ and it intersects the $\bar{B}$-axis at K_B. Thus the two isoclines (8.7a) and (8.7b) intersect at a unique point $(\bar{B},\bar{N})$ [see fig. 8.1] provided

$$\frac{1}{\bar{B}} + \frac{g'(\bar{B})}{g(\bar{B})} \;<\; \frac{p'(\bar{B})}{p(\bar{B})} \qquad (8.10a)$$

and $\qquad \dfrac{g_0}{\beta\, p'(0)} \;>\; K_0 \qquad\qquad\qquad\qquad (8.10b)$

for $\quad 0 < B \le K_B$

Hence the equilibrium $E_5(\bar{N},\bar{B},0)$ exists provided (8.10) holds.

Existence of $E_6(0,\tilde{B},\tilde{P})$:

Here $\tilde{B}$ and $\tilde{P}$ are given by

$$\tilde{P} = \frac{g(\tilde{B})}{\alpha} \qquad\qquad\qquad\qquad\qquad (8.11a)$$

$$\tilde{P} = L\left(1 + \frac{\alpha_1\, \tilde{B}}{r_P}\right) \qquad\qquad\qquad (8.11b)$$

From (8.11a) we have

$$\lim_{\tilde{B}\to K_B} \tilde{P} = 0\;, \quad \lim_{\tilde{B}\to 0} \tilde{P} = \frac{g_0}{\alpha} \quad \text{and} \quad \frac{d\tilde{P}}{dB} = \frac{g'(\tilde{B})}{\alpha} < 0 \qquad (8.12)$$

and from (8.11b),

$$\lim_{\tilde{B}\to 0} \tilde{P} = L \quad \text{and} \quad \frac{d\tilde{P}}{d\tilde{B}} = \frac{\alpha_1\, L}{r_P} > 0 \qquad\qquad (8.13)$$

From (8.11a) and (8.12) we note that $\tilde{P}$ is a decreasing function of $\tilde{B}$ starting from (g_0/α) and the isocline (8.11a) intersects $\tilde{B}$-axis at K_B. From (8.11b) and (8.13) we note that $\tilde{P}$ is an increasing function of $\tilde{B}$ starting from L. Thus the two isoclines (8.11a) and (8.11b) intersect at a unique point $(\tilde{B},\tilde{P})$ [see fig. 8.2] provided

$$g_0 > \alpha L \qquad (8.14)$$

Hence the equilibrium $E_6(0,\tilde{B},\tilde{P})$ exists provided (8.14) holds.

Existence of $E^*(N^*,B^*,P^*)$:

Here N^*, B^* and P^* are given by

$$N^* = \frac{r(B^*) \; K(P^*)}{r_0} \qquad (8.15a)$$

$$\alpha \, P^* = g(B^*) - \frac{\beta \; r(B^*) \; K(P^*) \; p(B^*)}{r_0 \; B^*} \qquad (8.15b)$$

$$P^* = L \left(1 + \frac{\alpha_1 \; B^*}{r_P} \right) \qquad (8.15c)$$

From (8.15a) we note that N^* decreases as B^* decreases or as P^* increases. For the existence of E^*, it suffices to show that the isocline (8.15b) and (8.15c) intersect. As before from (8.15c) we note that P^* is an increasing function of B^* starting from L.

From (8.15b) we note the following points :

When $B^* \to 0$, $P^* \to P_c^*$ where P_c^* is given by

$$\alpha \, P_c^* = g_0 - \beta \; K(P_c^*) \; p'(0) \qquad (8.16a)$$

For P_c^* to be positive, we must have

$$\frac{g_0}{p'(0)} > \beta \; K_0 \qquad (8.16b)$$

Also when $B^* \to K_B$, $P^* \to P_k^*$ where P_k^* is given by

$$P_k^* = - [\; \beta \; r(K_B) \; K(P_k^*) \; p(K_B) \;]/ \; \alpha \; r_0 \; K_B < 0 \qquad (8.16c)$$

Equation (8.15b) also gives

$$\left[\alpha + \frac{\beta\, r(B^*)\, K'(P^*)\, p(B^*)}{r_0\, B^*} \right] \frac{dP^*}{dB^*}$$

$$= g'(B^*) - [\, \beta\, r'(B^*)\, K(P^*)\, p(B^*)\,]/\, r_0\, B^*$$

$$- \beta\, r(B^*)\, K(P^*)\, [\, B^*\, p'(B^*) - p(B^*)\,]/\, r_0\, B^{*2}$$

$$= F_1 \quad \text{(say)} \tag{8.16d}$$

Let $F_2 = \alpha + \dfrac{\beta\, r(B^*)\, K'(P^*)\, p(B^*)}{r_0\, B^*}$ \hfill (8.16e)

From (8.16d) we note that $\dfrac{dP^*}{dB^*}$ may be positive or negative depending upon the value of the functions $r(B)$, $p(B)$ and $K(P)$. If we assume that

$$\text{either} \quad \text{(i)} \quad F_1 < 0, \quad F_2 > 0 \tag{8.17a}$$

$$\text{or} \quad \text{(i)} \quad F_1 > 0, \quad F_2 < 0 \tag{8.17b}$$

holds, then $\dfrac{dP^*}{dB^*}$ is always negative. Thus the isocline (8.15b) is a decreasing function of P^* starting from P_c^* under the condition (8.16b) and (8.17), and the isocline (8.15c) is an increasing function of P^* starting from L. Hence the isoclines (8.15b) and (8.15c) intersect at a unique point, provided

$$0 < L < P_c^* \tag{8.18}$$

The intersection value of the above isoclines gives the $B^* - P^*$ coordinates of E^*. Knowing the value of B^* and P^*, N^* can be computed from (8.15a). Thus, the equilibrium E^* exists if inequalities (8.16b), (8.17) and (8.18) hold.

The local stability of equilibria can be studied from the following variational matrix which is obtained from (8.1) as follows:

$$M = \begin{bmatrix} r(B) - \dfrac{2\,r_0\,N}{K(P)} & r'(B)\,N & \dfrac{r_0\,N^2\,K'(P)}{K^2(P)} \\[2mm] -\beta\,p(B) & H & -\alpha\,B \\[2mm] 0 & \alpha_1\,P & G \end{bmatrix}$$

where $H = B\,g'(B) + g(B) - \beta\,N\,p'(B) - \alpha\,P$ (8.19a)

$$G = r_P - \frac{2\,r_P\,P}{L} + \alpha_1\,B \tag{8.19b}$$

Now, utilizing analogous notation for the equilibria (i.e. M_0 is the variational matrix corresponding to E_0), we obtain

$$M_0 = \begin{bmatrix} r_0 & 0 & 0 \\ 0 & g_0 & 0 \\ 0 & 0 & r_P \end{bmatrix}$$

$$M_1 = \begin{bmatrix} -r_0 & r'(0)\,K_0 & r_0\,K'(0) \\ 0 & g_0 - \beta\,K_0\,p'(0) & 0 \\ 0 & 0 & r_P \end{bmatrix}$$

$$M_2 = \begin{bmatrix} r(K_B) & 0 & 0 \\ -\beta\,p(K_B) & K_B\,g'(K_B) & -\alpha\,K_B \\ 0 & 0 & r_P + \alpha_1 K_B \end{bmatrix}$$

$$M_3 = \begin{bmatrix} r_0 & 0 & 0 \\ 0 & g_0 - \alpha\,L & 0 \\ 0 & \alpha_1\,L & -r_P \end{bmatrix}$$

$$M_4 = \begin{bmatrix} -r_0 & r'(0) \ K(L) & r_0 \ K'(L) \\ 0 & g_0 - \beta \ K(L) \ p'(0) - \alpha \ L & 0 \\ 0 & \alpha_1 \ L & -r_P \end{bmatrix}$$

$$M_5 = \begin{bmatrix} -r(\bar{B}) & r'(\bar{B}) \ \bar{N} & r^2(\bar{B}) \ K'(0)/r_0 \\ -\beta \ p(\bar{B}) & \bar{H} & -\alpha \ \bar{B} \\ 0 & 0 & r_P + \alpha_1 \ \bar{B} \end{bmatrix}$$

where $\bar{H} = [\bar{B}^2 \ g'(\bar{B}) - \beta \ \bar{N} \ \{ B \ p'(\bar{B}) - p(\bar{B}) \}]/ \ \bar{B}$ (8.20)

$$M_6 = \begin{bmatrix} r(\tilde{B}) & 0 & 0 \\ -\beta \ p(\tilde{B}) & \tilde{B} \ g'(\tilde{B}) & -\alpha \ \tilde{B} \\ 0 & \alpha_1 \ \tilde{P} & -(r_P \ \tilde{P})/L \end{bmatrix}$$

$$M^* = \begin{bmatrix} -r(B^*) & r'(B^*) \ N^* & r^2(B^*) \ K'(P^*)/r_0 \\ -\beta \ p(B^*) & H^* & -\alpha \ B^* \\ 0 & \alpha_1 \ P^* & -(r_P \ P^*)/L \end{bmatrix}$$

where $H^* = [B^{*2} \ g'(B^*) - \beta \ N^* \{B^* \ p'(B^*) - p(B^*)\}]/B^*$ (8.21)

From above the variational matrices and standard stability theory of ordinary differential equation, we note the following obvious remarks: E_0 is unstable in N-B-P plane. E_1 is a saddle point whose stable manifold is locally in N-direction and whose unstable manifold is in B-P plane. E_2 is also a saddle point whose stable manifold is locally in B-direction and whose unstable manifold is in N-P plane. Again E_3 is a saddle point with stable

manifold in P-direction and unstable manifold in N-B plane. E_4 is also a saddle point whose stable manifold is locally in N-P plane and whose unstable manifold is in B direction (Note: g_0 - $\beta K(L)p'(0)$ - αL is taken positive in this case). From M_5, we note that E_5 is locally unstable in P direction. Further, using Routh-Hurwitz criterion it can be checked that E_5 is locally stable in N-B plane, provided

$$\bar{H} < 0 \tag{8.22}$$

Similarly from M_6, we note that E_6 is locally unstable in N direction and locally stable in B-P plane.

In general there is no obvious remarks to be made about the stability of E^*. However to find a sufficient condition for E^* to be locally asymptotically stable, we prove the following theorem.

THEOREM 8.2.1 Let the following inequalities hold:

$$H^* < 0 \tag{8.23a}$$

$$r(B^*) > \beta \, p(B^*) \tag{8.23b}$$

$$- H^* > r'(B^*) \, N^* + \alpha_1 \, P^* \tag{8.23c}$$

$$\frac{r_P \, P^*}{L} > \alpha \, B^* - \frac{r^2(B^*) \, K'(P^*)}{r_0} \tag{8.23d}$$

then E^* is locally asymptotically stable.

Proof: If inequalities (8.23) hold, then by Gershgorin's theorem (Lancaster and Tismanetsky, 1985, p.371) all eigen values of M^* have negative real parts and the theorem follows.

In the following we will find conditions which guarantee that E^* is globally (asymptotically) stable. We first prove the following lemma which establishes the region of attraction for our system. The ideas used here are developed in Hsu (1978) and

Freedman (1987).

LEMMA 8.2.1   The set

$$\mathbb{R} = \{(N, B, I) : 0 \leq N \leq K_c, \quad 0 \leq B \leq K_B, \quad 0 \leq P \leq L_0)\}$$

is a region of attraction for all solutions initiating in the positive octant under the conditions (8.2) - (8.5), where

$$K_c = r(K_B) \, K_0/r_0, \quad L_0 = L\left(1 + (\alpha_1 \, K_B/r_P)\right).$$

Proof: Let $N(t)$, $B(t)$, $P(t)$ be any solutions with nonnegative initial conditions (N_0, B_0, P_0). Since

$$\frac{dB}{dt} = B \, g(B) - \beta \, N \, p(B) - \alpha \, P \, B$$

$$\leq B \, g(B)$$

$$\leq B \, g(0) \, (1 - \frac{B}{K_B}) \,, \text{ From (8.4) and mean value}$$

theorem [Note fig. 8.3 also]

hence $\lim\limits_{t \to \infty} B(t) \leq K_B$

Again we have

$$\frac{dN}{dt} = r(B)N - \frac{r_0 N^2}{K(P)}$$

$$\leq r(K_B) \, N \, (1 - \frac{N}{K_c})$$

hence $\lim\limits_{t \to \infty} N(t) \leq K_c$

We also have

$$\frac{dP}{dt} = r_I(1 - \frac{P}{L})P + \alpha_1 \, P \, B$$

$$\leq r_I \, P + \alpha_1 \, P \, K_B - \frac{r_P \, P^2}{L}$$

$$= (r_P + \alpha_1 \, K_B) \, I \, (1 - \frac{P}{L_o})$$

hence $\lim\limits_{t \to \infty} P(t) \leq L_0$

proving the lemma.

We now show that E^* is globally asymptotically stable under certain conditions by proving the following theorem.

THEOREM 8.2.2   In addition to the assumptions (8.2a), (8.3) – (8.5), let $r(B)$, $p(B)$, $g(B)$ and $K(P)$ satisfy in $\mathbb{R}$ the following conditions

$$K_m \leq K(P) \leq K_0 \ , \ 0 \leq K'(P) \leq k_m$$

$$0 \leq r'(B) \leq r_m \ , \ \rho_0 \leq - g'(B) \leq \rho_m \qquad (8.24)$$

$$p_1 \leq \frac{p(B)}{B} \leq p_2 \ , \ P_0 \leq \frac{d}{dB}\left(\frac{p(B)}{B}\right) \leq P_m$$

for some positive constants K_m, k_m, r_m, ρ_0, ρ_m, P_0, P_m, p_1, p_2. If following inequalities hold

$$\left(r_m + \frac{\alpha_1 \ \beta \ p_2}{\alpha} \right)^2 < \frac{2 \ \alpha_1 \ r_0}{\alpha \ K(P^*)} \ (\beta \ p_0 \ N^* + \rho_0) \qquad (8.25a)$$

$$\frac{K_c^2 \ k_m^2}{K_m^4} < \frac{2 \ r_P}{r_0 \ L \ K(P^*)} \qquad (8.25b)$$

Then E^* is globally asymptotically stable with respect to all solutions initiating in the positive octant.

Proof: We consider the following positive definite function about E^*,

$$V(N,B,I) = N - N^* - N^* \ln(N/N^*) + c_1 \ (B - B^* - B^* \ln(B/B^*))$$

$$+ c_2 \ (P - P^* - P^* \ln(P/P^*)) \qquad (8.26)$$

where c_1 and c_2 are positive constants to be chosen later. Differentiating V with respect to t along the solutions of (8.1), we get

$$\frac{dV}{dt} = (N - N^*)\left[r(B) - \frac{r_0 N}{K(P)}\right] + c_1(B - B^*)\left[g(B) - \frac{\beta \ N \ p(B)}{B} - \alpha P\right]$$

$$+ c_2 \ (P - P^*)\left[r_P(1 - \frac{P}{L}) + \alpha_1 \ B\right] \qquad (8.27)$$

using (8.15) in (8.27) and simplifying we get

$$\frac{dV}{dt} = -\frac{r_0}{K(P^*)} (N - N^*)^2 - c_1[\beta N^* \mu(B) - \nu(B)](B - B^*)^2$$

$$- \frac{c_2 r_P}{L} (P - P^*)^2 + [\eta(B) - \frac{c_1 \beta p(B)}{B}](N - N^*)(B - B^*)$$

$$- r_0 N \xi(P) (N - N^*)(P - P^*)$$

$$+ (c_2 \alpha_1 - c_1 \alpha)(B - B^*)(P - P^*) \qquad (8.28)$$

where

$$\xi(P) = \begin{cases} \left(\dfrac{1}{K(P)} - \dfrac{1}{K(P^*)} \right)/(P - P^*), & P \neq P^* \\[2mm] -\dfrac{K'(P^*)}{K^2(P^*)}, & P = P^* \end{cases} \qquad (8.29a)$$

$$\eta(B) = \begin{cases} [r(B) - r(B^*)]/(B - B^*), & B \neq B^* \\[2mm] r'(B), & B = B^* \end{cases} \qquad (8.29b)$$

$$\nu(B) = \begin{cases} [g(B) - g(B^*)]/(B - B^*), & B \neq B^* \\[2mm] g'(B^*), & B = B^* \end{cases} \qquad (8.29c)$$

$$\mu(B) = \begin{cases} \left(\dfrac{p(B)}{B} - \dfrac{p(B^*)}{B^*} \right)/(B - B^*), & B \neq B^* \\[2mm] \dfrac{d}{dB}(\dfrac{p(B)}{B})\Big|_{B^*} & B = B^* \end{cases} \qquad (8.29d)$$

We note from (8.24) and the mean value theorem that

$$|\xi(P)| \leq k_m / K_m^2, \quad |\eta(B)| \leq r_m,$$

$$p_0 \leq \mu(B) \leq p_m, \quad \rho_0 \leq -\nu(B) \leq \rho_m. \qquad (8.30)$$

Now choosing the constant $c_1 = \dfrac{c_2 \alpha_1}{\alpha}$, $c_2 = 1$, $\dfrac{dV}{dt}$ cab further be written be written as the sum of the quadratics

$$\frac{dV}{dt} = -\frac{1}{2} a_{11} (N - N^*)^2 + a_{12} (N - N^*)(B - B^*) - \frac{1}{2} a_{22} (B - B^*)^2$$

$$-\frac{1}{2} a_{11} (N - N^*)^2 + a_{13} (N - N^*)(P - P^*) - \frac{1}{2} a_{33} (P - P^*)^2$$

$$(8.31)$$

where $a_{11} = \dfrac{r_0}{K(P^*)}$, $a_{22} = \dfrac{2 \alpha_1}{\alpha} [\beta N^* \mu(B) - \nu(B)]$

$a_{33} = \dfrac{2 r_P}{L}$, $a_{12} = \eta(B) - \dfrac{\alpha_1 \beta p(B)}{\alpha B}$, $a_{13} = - r_0 N \xi(P)$

Then a sufficient condition for $\frac{dV}{dt}$ to be negative definite is that the following inequalities hold:

$$a_{12}^2 - a_{11} a_{22} < 0 \qquad\qquad (8.32a)$$

$$a_{13}^2 - a_{11} a_{33} < 0 \qquad\qquad (8.32b)$$

Since (8.25a) ⇒ (8.32a) and (8.25b) ⇒ (8.32b), we conclude that V is a Liapunov function with respect to E^* whose domain contains the region R. Thus proving the theorem.

The above theorem implies that if human and cattle populations increase without control, then the density of the resource biomass decreases leading to the decrease of wildlife species and its survival may be threatened if the population rise continues.

CASE II: THE WILDLIFE SPECIES BEING WHOLLY DEPENDENT ON THE RESOURCE

In this case, the model can be analysed in the similar fashion as discussed in case I. For example, when r(0) = 0, it can be seen that there are only six equilibria, namely $E_0(0,0,0)$, $E_2(0,K_B,0)$, $E_3(0,0,L)$, $E_5(\bar{N},\bar{B},0)$, $E_6(0,\tilde{B},\tilde{P})$, and $E^*(N^*,B^*,P^*)$. It should be noted here that the equilibria corresponding to E_1 and E_4 do not exist in this case which shows that the wildlife species can not survive without resource. It can also be checked that E_5

exists under the condition (8.10a) only and the condition (8.10b) is satisfied automatically . E_6 exists under the same condition (8.14) and E^* exists under the same conditions (8.16b), (8.17) and (8.18). The stability behavior of the above equilibria is similar to the corresponding equilibria as discussed in case I.

CASE III: THE WILDLIFE SPECIES ACTING AS A PREDATOR ON THE RESOURCE

Again in the case, the model can be analysed in the similar way as the case I. For example, it can be checked that there are only six equilibria $E_0(0,0,0)$, $E_2(0,K_B,0)$, $E_3(0,0,L)$, $E_5(\bar{N},\bar{B},0)$, $E_6(0,\tilde{B},\tilde{P})$, and $E^*(N^*,B^*,P^*)$. In this case also, the equilibria corresponding to E_1 and E_4 do not exist. E_5 exists if in addition to (8.10a), $B_a < K_B$ holds. E_6 exists under the same condition (8.14) and E^* exists under the same conditions (8.16b), (8.17) and (8.18). The stability behavior of these equilibria is similar to the corresponding equilibria as discussed in the case I.

8.3 SPECIAL CASE:

Taking
$$\left.\begin{array}{c} r(B) = r_0 + r_1 B \\ K(P) = K_0 - k_1 P \\ g(B) = r_B(1 - \dfrac{B}{K_B}) \\ p(B) = B \end{array}\right] \qquad (8.33)$$
and

where r_1, k_1, r_B are positive constants

our model (8.1) reduces to

$$\frac{dN}{dt} = (r_0 + r_1 B)N - \frac{r_0 N^2}{K_0 - k_1 P}$$

$$\frac{dB}{dt} = r_B(1 - \frac{B}{K_B})B - \beta N B - \alpha P B \qquad (8.34)$$

$$\frac{dP}{dt} = r_P(1 - \frac{P}{L})P + \alpha_1 P B$$

The model (8.34) can be analysed as in the general case and the corresponding theorems can be obtained. For example, for the positive equilibrium $E^*(N^*, B^*, P^*)$; N^*, B^* and P^* are given by

$$N^* = \frac{1}{r_0}(K_0 - k_1 P^*)(r_0 + r_1 B^*) \tag{8.35a}$$

$$\alpha P^* = r_B(1 - \frac{B^*}{K_B}) - \frac{\beta}{r_0}(r_0 + r_1 B^*)(K_0 - k_1 P^*) \tag{8.35b}$$

$$P^* = L(1 + \frac{\alpha_1 B^*}{r_P}) \tag{8.35c}$$

It can be easily checked that the equilibrium E^* exists if the following inequalities hold :

$$\alpha r_0 > \beta k_1(r_0 + r_1 B^*) \tag{8.36a}$$

$$r_B > \beta K_0, \quad \alpha > \beta k_1 \tag{8.36b}$$

$$0 < L < P_c^* \tag{8.36c}$$

for $0 < B^* \leq K_B$

where $P_c^* = (r_B - \beta K_0)/(\alpha - \beta k_1)$

It can be easily proved that the equilibrium E^* is locally asymptotically stable provided the following inequalities hold

$$r_0 > (\beta - r_1) B^* \tag{8.37a}$$

$$r_B B^* > K_B (\alpha_1 P^* + r_1 N^*) \tag{8.37b}$$

$$\frac{r_P P^*}{L} > \alpha B^* + \frac{r_0 N^{*2} k_1}{(K_0 - k_1 P^*)^2} \tag{8.37c}$$

We also note that the region of attraction for the model (8.34) is same as that of the model (8.1).

It can also be checked that if $K(P) = K_0 - k_1 P$ satisfy in $\mathbb{R}$

$$K_m \leq K_0 - k_1 P \leq K_0 \quad \text{and} \quad 0 \leq k_1 \leq k_m \tag{8.38}$$

for some positive constants K_m, k_m and if the following

inequalities hold

$$\left(r_1 + \frac{\alpha_1 \beta}{\alpha}\right)^2 < \frac{2 \, r_0 \, r_B \, \alpha_1}{K_B(K_0 - k_1 I^*)\alpha} \qquad (8.39a)$$

$$\frac{K_c^2 \, k_m^2}{K_m^4} < \frac{2 \, r_0 \, r_P}{r_0 \, L(K_0 - k_1 P^*)} \qquad (8.39b)$$

then E^* is globally asymptotically stable with respect to all solutions initiating in the positive octant. Its proof is under same line as that of theorem 8.3.2. Using the same Liapunov function (8.26) and choosing $c_1 = \alpha_1/\alpha$, $c_2 = 1$, it can be checked that in this case

$$a_{11} = \frac{r_0}{K_0 - k_1 P^*} \, , \quad a_{22} = 2 \, \alpha \, r_B/\alpha_1 \, K_B \, , \quad a_{33} = 2 \, r_P/L$$

$$a_{12} = r_1 - \frac{\alpha_1 \beta}{\alpha} \, , \quad a_{13} = - \, r_0 \, N \, \xi(P) \, ,$$

$$\xi(P) = \begin{cases} \left[\dfrac{1}{K_0 - k_1 P} - \dfrac{1}{K_0 - k_1 P^*}\right] \Big/ (P - P^*) & P \neq P^* \\[2em] - \dfrac{k_1}{(K_0 - k_1 P^*)^2} & P = P^* \end{cases}$$

$$\eta(B) = r_1 \, , \quad \nu(B) = - \, r_B/K_B \, , \quad \mu(B) = 0.$$

8.4 CONSERVATION MODEL

In the previous section section it is noted that if the cumulative density of human and cattle populations increases, then the density of the resource decreases and consequently the survival of the resource dependent wildlife species may be threatened. Therefore some efforts must be applied to conserve the resource and to control the cumulative density of human and cattle

populations. Let $F_1(t)$ be the density of effort applied to conserve the resource and $F_2(t)$ be the density of effort applied to control the cumulative density of human and cattle populations. It is assumed that $F_1(t)$ is proportional to the depleted level of biomass from its carrying capacity and $F_2(t)$ is proportional to undesired level of the density of human and cattle populations. Then the dynamics of the system can be written as

$$\frac{dN}{dt} = r(B)N - \frac{r_0 N^2}{K(P)}$$

$$\frac{dB}{dt} = B\, g(B) - \beta\, N\, p(B) - \alpha\, P\, B + r_{10} F_1$$

$$\frac{dP}{dt} = r_P\left(1 - \frac{P}{L}\right)P + \alpha_1\, P\, B - r_{20} F_2 P \qquad (8.40)$$

$$\frac{dF_1}{dt} = r_1\left(1 - \frac{B}{K_B}\right) - \nu_1 F_1$$

$$\frac{dF_2}{dt} = r_2(P - P_c) - \nu_2 F_2$$

$N(0) = N_0 \geq 0$, $B(0) = B_0 \geq 0$, $P(0) = P_0 \geq 0$, $F_1(0) = F_{i0} \geq 0$,
$i = 1, 2$.

Here r_1, r_2 are the growth rate coefficients of $F_1(t)$, $F_2(t)$ respectively and ν_1, ν_2 are their respective depletion rate coefficients. r_{10} is the growth rate coefficient of the resource biomass $B(t)$ due to the effort $F_1(t)$ and r_{20} is depletion rate coefficient of human and cattle population $P(t)$ due to the effort $F_2(t)$.

Here, we analyse the model (8.40) only for the case when $r(B)$ satisfies the condition (8.2a).

It can be checked that there are two equilibria, namely $\tilde{E}(0, \tilde{B}, \tilde{P}, \tilde{F}_1, \tilde{F}_2)$ and $\hat{E}(\hat{N}, \hat{B}, \hat{P}, \hat{F}_1, \hat{F}_2)$. Here $\tilde{B}$, $\tilde{P}$, $\tilde{F}_1$ and $\tilde{F}_2$ are the positive solution of the system of following algebraic equations:

$$\alpha P = g(B) + \frac{r_1 r_{10}}{\nu_1 B} - \frac{r_1 r_{10}}{\nu_1 K_B} \tag{8.41a}$$

$$P = \frac{L\ [r_p \nu_2 + r_2 r_{20} P_c + \alpha_1 \nu_2 B]}{r_p \nu_2 + r_2 r_{20} L} \tag{8.41b}$$

$$F_1 = \frac{r_1}{\nu_1}\ (1 - \frac{B}{K_B}) \tag{8.41c}$$

$$F_2 = \frac{r_2}{\nu_2}\ (P - P_c) \tag{8.41d}$$

It is easy to check that the isoclines (8.41a) and (8.41b) intersect at a unique point. The intersection value of these two isoclines gives the B-P coordinates of $\tilde{E}$ and its other coordinates can be determined from (8.41c) and (8.41d).

Further the coordinates $\hat{N}$, $\hat{B}$, $\hat{P}$, $\hat{F}_1$ and $\hat{F}_2$ of $\hat{E}$ are the positive solution of the following system of equations:

$$N = \frac{r(B)K(P)}{r_0} \tag{8.42a}$$

$$\alpha P = g(B) - \frac{\beta r(B)K(P)p(B)}{r_0 B} + \frac{r_1 r_{10}}{\nu_1 B} - \frac{r_1 r_{10}}{\nu_1 K_B} \tag{8.42b}$$

$$P = \frac{L\ [r_p \nu_2 + r_2 r_{20} P_c + \alpha_1 \nu_2 B]}{r_p \nu_2 + r_2 r_{20} L} \tag{8.42c}$$

$$F_1 = \frac{r_1}{\nu_1}\ (1 - \frac{B}{K_B}) \tag{8.42d}$$

$$F_2 = \frac{r_2}{\nu_2}\ (P - P_c) \tag{8.42e}$$

Again it can be checked that the isoclines (8.42b) and (8.42c) will intersect at a unique point, provided

$$\text{either (i) } G_1 < 0,\ G_2 > 0 \tag{8.43a}$$

$$\text{or} \quad \text{(ii) } G_1 > 0,\ G_2 < 0 \tag{8.43b}$$

holds, where

$$G_1 = \alpha + \frac{\beta r(B)K'(P)p(B)}{r_0 B}$$

$$G_2 = g'(B) - \frac{\beta r'(B)K(P)p(B)}{r_0 B} - \frac{\beta r(B)K(P)}{r_0 B^2} [B\, p'(B) - p(B)] - \frac{r_1 r_{10}}{\nu_1 B^2}$$

By computing the variational matrix corresponding to $\overset{\approx}{E}$, it can be checked that $\overset{\approx}{E}$ is a unstable point. In the following theorem we have found the conditions for $\hat{E}$ to be locally asymptotically stable whose proof is similar to theorem 8.3.1.

THEOREM 8.4.1 Let the following inequalities hold

$$\hat{H} < 0 \qquad\qquad (8.44a)$$

$$r(\hat{B}) > \beta\, p(\hat{B}) \qquad\qquad (8.44b)$$

$$-\hat{H} > r'(\hat{B})\,\hat{N} + \alpha_1 \hat{P} + \frac{r_1}{K_B} \qquad\qquad (8.44c)$$

$$\frac{r_P \hat{P}}{L} > \alpha\, \hat{B} - \frac{r^2(\hat{B})\, K'(\hat{P})}{r_0} + r_2 \qquad\qquad (8.44d)$$

$$\nu_1 > r_{10} \qquad\qquad (8.44e)$$

$$\nu_2 > r_{20}\hat{P} \qquad\qquad (8.44f)$$

Then $\hat{E}$ is locally asymptotically stable, where

$$\hat{H} = \hat{B}\, g'(\hat{B}) - \frac{\beta\hat{N}}{\hat{B}} [\hat{B}\, p'(\hat{B}) - p(\hat{B})] - \frac{r_{10}\hat{F}_1}{\hat{B}}$$

The following lemma establishes the region of attraction for the system (8.40) whose proof is similar to lemma 8.3.1.

LEMMA 8.4.1 The set
$$\hat{\mathbb{R}} = \left\{ (N,B,P) : 0 \le N \le K_a,\ 0 \le B \le K_b,\ 0 \le P \le L_a,\ 0 \le F_1 \le r_1/\nu_1, \right.$$
$$\left. 0 \le F_2 \le r_2 L_a/\nu_2 \right\} \text{ is a region of attraction for all}$$

solutions initiating in the positive octant, where
$$K_a = r(K_b)\, K_0/r_0, \quad K_b = \frac{K_B}{2}\left[1 + \left\{1 + \frac{4r_1 r_{10}}{\nu_1 g(0)K_B}\right\}^{1/2}\right],$$
$$L_a = L\left(1 + (\alpha_1 K_b/r_P)\right).$$

In the following theorem we have shown that $\hat{E}$ is globally asymptotically stable under certain conditions.

THEOREM 8.4.2   In addition to the assumptions (8.2a), (8.3) – (8.5), let $r(B)$, $p(B)$, $g(B)$ and $K(P)$ satisfy in $\hat{R}$ the following conditions

$$\hat{K}_m \le K(P) \le K_0 \ , \ 0 \le K'(P) \le \hat{k}_m$$

$$0 \le r'(B) \le \hat{r}_m \ , \ \hat{\rho}_0 \le - g'(B) \le \hat{\rho}_m \qquad (8.45)$$

$$\hat{p}_1 \le \frac{p(B)}{B} \le \hat{p}_2 \ , \ \hat{p}_0 \le \frac{d}{dB}\left(\frac{p(B)}{B}\right) \le \hat{p}_m$$

for some positive constants $\hat{K}_m$, $\hat{k}_m$, $\hat{r}_m$, $\hat{\rho}_0$, $\hat{\rho}_m$, $\hat{p}_0$, $\hat{p}_m$, $\hat{p}_1$, $\hat{p}_2$. Then if following inequalities hold

$$\left(\hat{r}_m + \frac{\alpha_1 \beta \hat{p}_2}{\alpha}\right)^2 < \frac{2 \alpha_1 r_0}{\alpha K(\hat{P})} (\beta \hat{p}_0 \hat{N} + \hat{\rho}_0) \qquad (8.46a)$$

$$\frac{K_a^2 \hat{k}_m^2}{\hat{K}_m^4} < \frac{2 r_P}{r_0 L K(\hat{P})} \qquad (8.46b)$$

$\hat{E}$ is globally asymptotically stable with respect to all solutions initiating in the positive octant.

Proof: Taking the following positive definite function about $\hat{E}$,

$$W(N,B,P,F_1,F_2) = N - \hat{N} - \hat{N} \ln(N/\hat{N}) + \frac{\alpha_1}{\alpha} (B - \hat{B} - \hat{B} \ln(B/\hat{B}))$$

$$+ (P - \hat{P} - \hat{P} \ln(P/\hat{P})) + \frac{\alpha_1 r_{10} K_B}{2\alpha r_1 \hat{B}} (F_1 - \hat{F}_1)^2 + \frac{r_{20}}{r_2} (F_2 - \hat{F}_2)^2 \quad (8.47)$$

it can be checked that the derivative of W with respect to t along the solution of (8.40) is negative definite under the condition (8.46), proving the theorem.

The above theorem implies that by conserving the biomass and by controlling the cumulative density of human and cattle populations, the density of resource can be maintained at an appropriate level and the survival of the resource dependent

wildlife species may be guaranteed.

8.5 CONCLUSIONS

In this chapter, a mathematical model has been proposed to study the growth and existence of resource (biomass) dependent wildlife species in a forested habitat such as Doon Valley in the northern province of Uttar Pradesh in India which is being continuously depleted due to growth of human and cattle populations and associated industrial development. It is assumed that growth rate of the density of wildlife population depends upon forest biomass but its carrying capacity in the habitat decreases due to human and cattle population density. It is also considered that the growth of human and cattle populations increases due to forested biomass. Thus the depletion of forest biomass is caused both by wildlife species, and human and cattle populations.

By using stability analysis it has been shown that forest biomass density decreases as the density of human and cattle populations increases and it may not last long if this population pressure continues unabatedly. It has also been shown that the effect of this biomass decrease and the increase in human and cattle populations densities lead to lowering down the density of the resource dependent wildlife species or even to its eventual extinction if the population increase continues without control. A model to conserve the resource and to control human and cattle populations is also presented. It is shown here that if suitable efforts are made to conserve the resource and to control the cumulative density of human and cattle populations, then the survival of resource dependent wildlife species can be guaranteed.
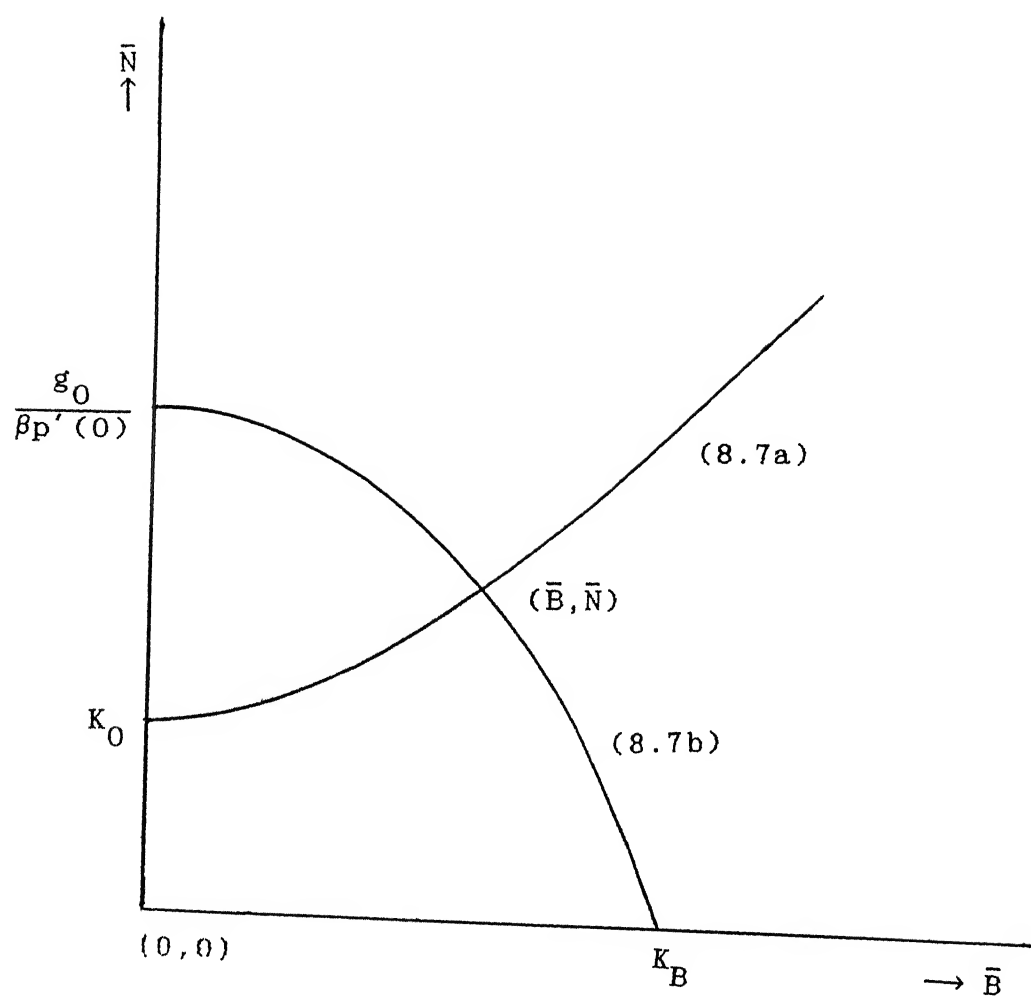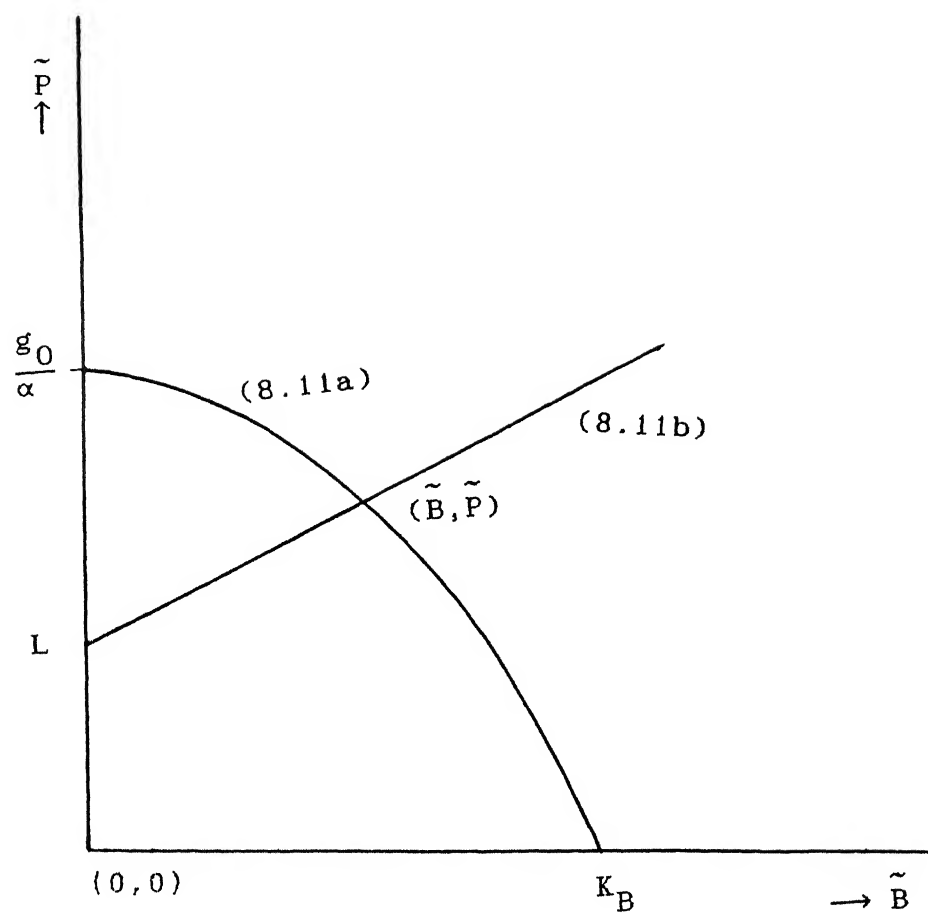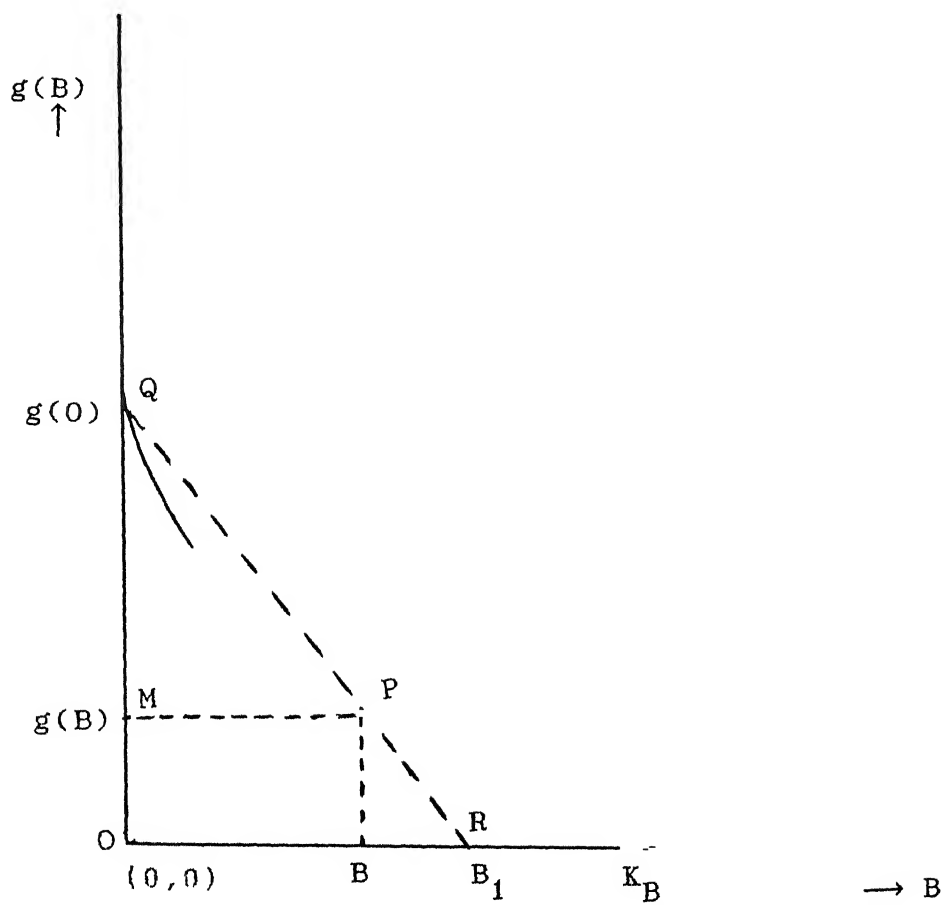
Fig. 8.1

Fig. 8.2

Fig. 8.3

Note from Δ^S QMP and QOR

$$\frac{g(0) - g(B)}{B} = \frac{g(0)}{B_1} > \frac{g(0)}{K_B} \text{ for } 0 < B \leq B_1 \leq K_B.$$

CHAPTER IX

EFFECT OF CHANGING HABITAT ON WILDLIFE SPECIES: APPLICATION TO KEOLADEO NATIONAL PARK, INDIA

9.0   INTRODUCTION

The ecosystems undergo degradation due to internal and external environmental influences causing significant changes in its structure and function, both aboitic and biotic(air, water, light, heat, plants, micro-organisms , animals, etc.). These changes may affect the biodiversity as well as the carrying capacity of the habitat. It may happen that one wild species in the habitat may grow uncontrolled affecting the growth and survival of other species. In the aquatic environment, a typical example of growth of noxious weeds (Eichhornia, Salvinia), which are harmful to other species, has been pointed out by many investigators, Thomas(1981), Webber(1978). Such changes in the habitat may adversely affect growth, migration and dispersion of certain other species, Cody and Diamond(1975), Connell(1978), Kormondy(1986), Luckinbill(1979), Whittaker(1967). Similar situation also exists in a large partly wetland Keoladeo National Park at Bharatpur(Rajasthan) in India. Here the habitat is degrading due to uncontrolled growth of wild grasses such as Paspalum distichum causing decrease in the population of phytoplancton, zooplankton, small and large fishes, birds, etc. In past decades, when buffaloes were allowed to graze the wild grasses, various species coexisted without any noticeable change, Ali and Vijayan(1986). In recent years buffaloes, for whom this grass is a stable food, have not been allowed to enter the Park

for grazing causing extensive increase in the density of the wild grasses in the far flung areas of the Park making the habitat unfavorable to various species. It has been pointed by Ali and Vijayan (1986) in their study on Keoladeo National Park that the biomass of the aquatic macrophytes has increased from 424 to 905 gm/m^2 between March 1983 to October 1985 [see fig.9.1, Ali and Vijayan (1986) p.47]. This biomass increase was caused mainly by four species, namely Paspalum distichum, Psuedoraphis spinescens, Ipomoea aquatic and Hydrilla verticillata and among these Paspalum distichum constituted the major portion of the total biomass. It has been pointed out that cessation of cattle grazing since November 1982 has increased the density of Paspalum distichum causing several changes in the habitat especially in the waterspread area as mentioned below, Ali and Vijayan(1986) :

(i) the extensive growth of Paspalum distichum has reduced the open water space,

(ii) the growth of submersed and floating vegetation has dwindled due to excessive growth of Paspalum distichum and species such as Nymphoides indicum, Nymphoides cristatum, Nymphaea nouchali and Nymphaea stellata are the most affected ones,

(iii) the production of oxygen by phytoplankton is reduced due to uncontrolled growth of Paspalum distichum,

(iv) the density of fish population is found to be less in the habitat where the biomass of the Paspalum distichum was more,

(v) the uncontrolled growth of Paspalum distichum has adversely affected waterfowl population in that region,

(vi) the arrival of Siberian cranes have also been affected due to excessive growth of Paspalum distichum due to which cranes felt

uncomfortable in that region and they shifted to the feeding areas,

(vii) the breeding populations of heronary species have also been reduced due to unchecked spreading and growth of Paspalum distichum.

Other species such as cyperus alopecuroides(a sedge) has also multiplied, invaded and colonized open waters, endangering the habitats of diving ducks and coots. Vetiveria zizanioides has spread rapidly into the waterspread area threatening the feeding habitats of the Siberian cranes.

It has been thus pointed out that the wetland area of the Park is being slowly and slowly converted into grassland – woodland biotype due to excessive growth of the amphibious grass(Paspalum distichum).

Keeping in view the above in this chapter, we propose a mathematical model to study the effect of degradation of habitat caused by the uncontrolled growth of wild grasses such as Paspalum distichum on growth and existence of other species living in the same habitat. Stability analysis, La Salle and Lefschetz(1961), is used to analyse the model to predict the behavior of the system.

9.1 MATHEMATICAL MODEL

As described above we divide the ecosystem in the wetland park into three components, the species of wild grasses, the wildlife species such as flora and fauna affected by the growth of the wild grasses and the buffaloes population allowed to enter into the park from outside for grazing the wild grasses in the park. Let $R(t)$ be the cumulative density of wildlife species in the habitat, $B(t)$ be the cumulative density of the wild grasses

and $P(t)$ be the density of buffaloes allowed to enter into the park for grazing at time t. We assume that the growth rate $r(B)$ of wildlife species and the corresponding carrying capacity $K(B)$ decrease with the increase in cumulative density $B(t)$ of wild grasses. We further assume that the growth rate of buffaloes allowed to enter into the park is proportional to $(B - B_c)$, the undesirable level of grass density over a critical value B_c not harmful to wildlife species. We also assume that the growth rate $r_B(P)$ of the wild grasses and its carrying capacity $K_B(P)$ decrease as the density $P(t)$ of the buffaloes population increases. With these considerations and assuming that $R(t)$ and $B(t)$ are governed by generalized logistic growth equation, the dynamics of the system can be written as follows :

$$\frac{dR}{dt} = r(B)R - \frac{r_0 R^2}{K(B)}$$

$$\frac{dB}{dt} = r_B(P)B - \frac{r_{B0} B^2}{K_B(P)} \tag{9.1}$$

$$\frac{dP}{dt} = \delta_1 (B - B_c) - \delta_0 P$$

$$R(0) \geq 0, \quad B(0) \geq 0, \quad P(0) \geq 0.$$

In the case, when buffaloes are not allowed to enter inside the park the third equation of (9.1) in the model can also be interpreted to govern the density of the labour force brought into the park to cut the wild grasses. In such a case the carrying capacity K_B of the wild grass may be a constant.

In our model (9.1), the function $r(B)$ represents the growth rate for $R(t)$ and it decreases as B increases and hence we assume

$$r(0) = r_0 > 0, \quad r'(B) < 0 \text{ for } B \geq 0,$$

and $\quad r(\bar{B}) = 0$ for some $\bar{B} > 0.$ $\tag{9.2}$

Also the function K(B) denotes the maximum population density of R(t) which the environment can support. It also decreases as B increases and hence we assume

$$K(0) = K_0 > 0, \quad K'(B) < 0, \quad \text{for } B \geq 0 \tag{9.3}$$

The above assumptions imply that the growth rate and carrying capacity of the wildlife species decrease considerably as the cumulative density of the wild grasses increases and if the density of the wild grasses is sufficiently high, then wildlife species can not grow and in fact may die out since $r(\bar{B}) = 0$ for some $\bar{B} > B_c > 0$.

Similarly the function $r_B(P)$ denotes the growth rate for B(t) and it satisfies

$$r_B(0) = r_{B0} > 0, \quad r_B'(P) < 0 \text{ for } P \geq 0$$

and $r_B(\bar{P}) = 0$ for some $\bar{P} > 0$ \hfill (9.4)

The function $K_B(P)$ represents the maximum density of B(t) which the environment can support and it also decreases as P increases. Hence we assume that

$$K_B(0) = K_{B0} > 0, \quad K_B'(P) < 0 \text{ for } P \geq 0 \tag{9.5}$$

In the model (9.1), δ_1 is the growth rate coefficient of P(t), δ_0 is the withdrawal or death rate coefficient of P(t) and B_c is the critical value of B which is harmless to wildlife species such as flora and fauna.

It should be noted here that if $B \leq B_c$, then $\frac{dP}{dt} < 0$ showing that there is no need to allow buffaloes inside the park for grazing. Hence through out our discussion we assume $B > B_c$.

9.2 MATHEMATICAL ANALYSIS

We first analyse the model when no grazing or cutting is allowed into the park. In such a case, the model (9.1) reduces to

$$\frac{dR}{dt} = r(B)R - \frac{r_0 R^2}{K(B)}$$

$$\frac{dB}{dt} = r_{BO}B - \frac{r_{BO}B^2}{K_{BO}}$$

(9.6)

$$R(0) \geq 0, \quad B(0) \geq 0.$$

In this case the model (9.6) has four nonnegative equilibria, namely $E_0(0.0)$, $E_1(K_0,0)$, $E_2(0,K_{BO})$ and $E_3(R_1,K_{Bo})$.
where

$$R_1 = r(K_{BO}) \, K(K_{BO})/r_0 \tag{9.7}$$

We note here that in E_3, R_1 decreases as K_{BO} increases and can tend to zero if $K_{BO} \geq \bar{B}$ [see equation (9.2)].

By computing the variational matrices corresponding to the equilibria we can check that E_0 is locally unstable in R-B plane, E_1 is a saddle point with stable manifold locally in R direction and with unstable manifold locally in B direction, E_2 is also a saddle point with stable manifold locally in B direction and with unstable manifold locally in R direction and E_3 is locally asymptotically stable in R-B plane.

To show that E_3 is globally asymptotically stable, we first need the following lemma which establishes the region of attraction for the system (9.6).

LEMMA 9.2.1   The set

$$\Omega_1 = \left\{ (R,B) : 0 \leq R \leq K_0, \; 0 \leq B \leq K_{BO} \right\} \text{ attracts all solutions}$$

initiating in the positive quadrant.

Proof: From (9.6) we have

$$\frac{dR}{dt} = r(B)R - \frac{r_0 R^2}{K(B)}$$

$$\leq r_0 R - r_0 R^2/K_0$$

hence $\lim_{t \to \infty} R(t) \leq K_0$

Similarly $\lim_{t \to \infty} B(t) \leq K_{BO}$

THEOREM 9.2.1   The equilibrium E_3 is globally asymptotically stable with respect to all solutions initiating in the interior of the positive quadrant.

Proof: We consider the following positive definite function about E_3,

$$V(R,B) = R - R_1 - R_1 \ln \frac{R}{R_1} + c(B - K_{BO} - K_{BO} \ln \frac{B}{K_{BO}}) \qquad (9.8)$$

where c is a positive constant to be chosen suitably.

In addition to the assumptions (9.2) and (9.3); let $r(B)$ and $K(B)$ satisfy in Ω_1

$$K_c \leq K(B) \leq K_0, \ 0 \leq - K'(B) \leq k_c, \ 0 \leq - r'(B) \leq r_c, \qquad (9.9)$$

for some positive constants K_c, k_c, r_c. By choosing

$$c > \left[\frac{r_0 k_c K_0}{K_c^2} + r_c \right]^2 \frac{K(K_{BO})}{4 r_0 r_{BO}} \qquad (9.10)$$

one can check that the time derivative of V along the solution of (9.6) is negative definite and hence V is a Liapunov function with respect to E_3 whose domain contains the region Ω_1, proving the theorem.

The above theorem implies that an increase in the cumulative density of wild grasses in the park will lower the population density of wildlife species. Further, if the carrying capacity K_{BO} of the wild grasses increases beyond a critical value $\bar{B}$ (i. e. K_{BO}

$\geq \bar{B}$), then the wildlife species may be doomed to extinction.

Now we analyse our general model (9.1) as follows.

In this case, the model (9.1) has the following two nonnegative equilibria : $\tilde{E}(0, \tilde{B}, \tilde{P})$ and $E^*(R^*, B^*, N^*)$. We shall show the existence of these equilibria as follows.

Existence of $\tilde{E}(0, \tilde{B}, \tilde{P})$:

Here $\tilde{B}$ and $\tilde{P}$ are the positive solution of the system of the equations

$$B = r_B(P)\ K_B(P)/r_{B0} \tag{9.11}$$

$$P = \delta_1(B - B_c)/\delta_0 \tag{9.12}$$

We note that the isoclines (9.11) is a decreasing function of P starting from $\bar{P}$ and the isocline (9.12) is an increasing function of B starting from B_c. Hence the two isoclines (9.11) and (9.12) must intersect at a unique point $(\tilde{B}, \tilde{P})$ for $B_c < K_{B0}$.

Existence of $E^*(R^*, B^*, P^*)$:

Here R^*, B^* and P^* are the positive solution of the system of algebraic equations

$$R = r(B)\ K(B)/r_0 \tag{9.13}$$

$$B = r_B(P)\ K_B(P)/r_{B0} \tag{9.14}$$

$$P = \delta_1(B - B_c)/\delta_0 \tag{9.15}$$

The existence of E^* follows from the existence of $\tilde{E}$.

Now to study the local stability behavior of the equilibria, we compute the variational matrices corresponding to the each equilibrium. Using the analogous notations for the variational matrices i.e. $\tilde{M}$ is the variational matrix corresponding to $\tilde{E}$, we obtain

$$
\tilde{M} = \begin{bmatrix} r(\tilde{B}) & 0 & 0 \\[2ex] 0 & -r_B(\tilde{P}) & r_B'(\tilde{P})\tilde{B} + \dfrac{r_B^2(\tilde{P})}{r_{BO}} K_B'(\tilde{P}) \\[2ex] 0 & \delta_1 & -\delta_0 \end{bmatrix}
$$

$$
M^* = \begin{bmatrix} -r(B^*) & r'(B^*)R^* + \dfrac{r^2(B^*)}{r_0} K'(B^*) & 0 \\[2ex] 0 & -r_B(P^*) & r_B'(P^*)B^* + \dfrac{r_B^2(P^*)}{r_{BO}} K_B'(P^*) \\[2ex] 0 & \delta_1 & -\delta_0 \end{bmatrix}
$$

From $\tilde{M}$ we note that $\tilde{E}$ is a saddle point with stable manifold locally in B-P plane and with unstable manifold locally in R direction. From M^* we note that E^* is locally asymptotically stable in R-B-P plane.

In the following theorem we are able to find sufficient conditions under which E^* is globally asymptotically stable. To prove this theorem we first require the following lemma which establishes the region of attraction for the system (9.1). The ideas used here are developed in Hsu(1978) and Freedman(1987).

LEMMA 9.2.2   The set
$$
\Omega_2 = \left\{ (R,B,P) : 0 \le R \le K_0,\ 0 \le B \le K_{BO},\ 0 \le P \le \delta_1 K_{BO}/\delta_0 \right\}
$$
attracts all solutions initiating in the positive octant.

Proof: As before, we have
$$
\lim_{t \to \infty} R(t) \le K_0
$$
and
$$
\lim_{t \to \infty} B(t) \le K_{BO}
$$

We also have,

$$\frac{dP}{dt} = \delta_1 (B - B_c) - \delta_0 P$$

$$\leq \delta_1 K_{B0} - \delta_0 P$$

and hence $\lim\limits_{t \to \infty} P(t) \leq \delta_1 K_{B0}/\delta_0$, proving the lemma.

THEOREM 9.2.2   In addition to the assumptions (9.2) — (9.5), let $r(B)$, $K(B)$, $r_B(P)$, $K_B(P)$ satisfy in Ω_2

$$K_m \leq K(B) \leq K_0, \quad 0 \leq - K'(B) \leq k_m, \quad 0 \leq - r'(B) \leq r_m,$$

$$K_s \leq K_B(P) \leq K_{B0}, \quad 0 \leq - K_B'(P) \leq k_s, \quad 0 \leq - r_B'(P) \leq r_s, \qquad (9.16)$$

for some positive constants K_m, K_s, k_m, k_s r_m, r_s. Then if the following inequalities hold

$$\left[r_m + \frac{r_0 k_m K_0}{K_m^2} \right]^2 < 2 \frac{r_0}{K(B^*)} \frac{r_{B0}}{K_B(P^*)} \qquad (9.17a)$$

$$\left[r_s + \frac{r_{B0} k_s K_{B0}}{K_s^2} + \delta_1 \right]^2 < 2 \delta_0 \frac{r_{B0}}{K_B(P^*)} \qquad (9.18b)$$

E^* is globally asymptotically stable with respect to all solutions initiating in the interior of the positive octant.

Proof: We consider the following positive definite function about E^*,

$$W(R,B,P) = \left[R - R^* - R^* \ln \frac{R}{R^*} \right] + \left[B - B^* - B^* \ln \frac{B}{B^*} \right]$$

$$+ \frac{1}{2} (P - P^*)^2 \qquad (9.18)$$

Differentiating W with respect to t along the solution of (9.1), we get

$$\frac{dW}{dt} = (R - R^*) \left[r(B) - \frac{r_0 R}{K(B)} \right] + (B - B^*) \left[r_B(P) - \frac{r_{B0} B}{K_B(P)} \right]$$

$$+ (P - P^*)(\delta_1 B - \delta_1 B_c - \delta_0 P)$$

A little algebraic manipulation yields,

$$\frac{dW}{dt} = - \frac{r_0}{K(B^*)} (R - R^*)^2 - \frac{r_{BO}}{K_B(P^*)} (B - B^*)^2 - \delta_0 (P - P^*)^2$$

$$+ (R - R^*)(B - B^*)\Big[\eta(B) - r_0 R\xi(B)\Big]$$

$$+ (B - B^*)(P - P^*)\Big[\eta_B(P) - r_{BO} R\xi_B(P) + \delta_1\Big] \qquad (9.19)$$

where

$$\eta(B) = \begin{cases} [r(B) - r(B^*)]/(B - B^*), & B \neq B^* \\ \\ r'(B^*), & B = B^* \end{cases} \qquad (9.20)$$

$$\xi(B) = \begin{cases} \left[\frac{1}{K(B)} - \frac{1}{K(B^*)}\right]/(B - B^*), & B \neq B^* \\ \\ -\frac{K'(B^*)}{K^2(B^*)}, & B = B^* \end{cases} \qquad (9.21)$$

$$\eta_B(P) = \begin{cases} [r_B(P) - r_B(P^*)]/(P - P^*), & P \neq P^* \\ \\ r'_B(P^*), & P = P^* \end{cases} \qquad (9.22)$$

$$\xi_B(P) = \begin{cases} \left[\frac{1}{K_B(P)} - \frac{1}{K_B(P^*)}\right]/(P - P^*), & P \neq P^* \\ \\ -\frac{K'_B(P^*)}{K^2(B^*)}, & P = P^* \end{cases} \qquad (9.23)$$

Using (9.16) and mean value theorem, we note that

$$|\eta(B)| \leq r_m, \quad |\xi(B)| \leq k_m/K_m^2, \quad |\eta_B(P)| \leq r_s, \quad |\xi_B(P)| \leq k_s/K_s^2. \qquad (9.24)$$

Now $\frac{dW}{dt}$ can further be written as sum of the quadratics as

$$\frac{dW}{dt} = -\frac{1}{2} a_{11}(R - R^*)^2 + a_{12}(R - R^*)(B - B^*) - \frac{1}{2} a_{22}(B - B^*)^2$$

$$-\frac{1}{2} a_{22}(B - B^*)^2 + a_{23}(B - B^*)(P - P^*) - \frac{1}{2} a_{33}(P - P^*)^2 \qquad (9.25)$$

where

$$a_{11} = \frac{2r_0}{K(B^*)} \; , \; a_{22} = \frac{r_{B0}}{K_B(P^*)} \; , \; a_{33} = 2\delta_0$$

$$a_{12} = \eta(B) - r_0 R\xi(B), \; a_{23} = \eta_B(P) - r_{B0}R\xi_B(P) + \delta_1 .$$

The sufficient conditions for $\frac{dW}{dt}$ to be negative definite are that

$$a_{12}^2 < a_{11} a_{22} \tag{9.26a}$$

$$a_{23}^2 < a_{22} a_{33} \tag{9.26b}$$

hold. We note that (9.17a) $\Rightarrow$ (9.26a) and (9.17b) $\Rightarrow$ (9.27b), hence W is a Liapunov function with respect to E^*, whose domain contains the region of attraction Ω_2, proving the theorem.

The above theorem shows that the flora and fauna in the park can boom if the growth of wild grasses is controlled either by buffaloes grazing or cutting them by labour forces. It may be noted here that if large number of buffaloes are allowed they may affect the carrying capacity of the habitat corresponding to various other species. This aspect is not included in the model.

9.3 CONCLUSIONS

In this chapter a mathematical model is proposed to study the effect of degrading habitat on the survival of wildlife species living in that habitat. The model presented here is proposed keeping in view the changes that exist in Keoladeo National Park at Bharatpur (Rajasthan) in India due to over growth of wild grasses such as Paspalum distichum. The basic assumption in this chapter is that the cumulative growth rate density and the corresponding carrying capacity of the wildlife species such as flora and fauna decrease as the density of the wild grasses increases. It is also assumed that the growth rate density of

buffaloes allowed to enter into the park for grazing the wild grasses is proportional to the undesirable level of the density of the grass. It is considered further that the growth rate of the cumulative density of wild grasses and the corresponding carrying capacity decrease with the increase in the density of buffaloes population. The model is analysed in two different parts.

In the first part, when no buffalo is allowed inside the park, it is shown that the growth rate of wild grasses will lower the cumulative density of wildlife species such as flora and fauna. It is further noted here that if the wild grasses are allowed to grow uncontrolled then the wildlife species may be doomed to extinction.

In the second part, the model is studied when buffaloes are allowed inside the park for grazing the wild grasses. It is shown that the wildlife species can boom if the growth of wild grasses is controlled by buffaloes grazing or by cutting them using labour forces.

The model presented here is also applicable to other wetland parks around the world having similar ecology.
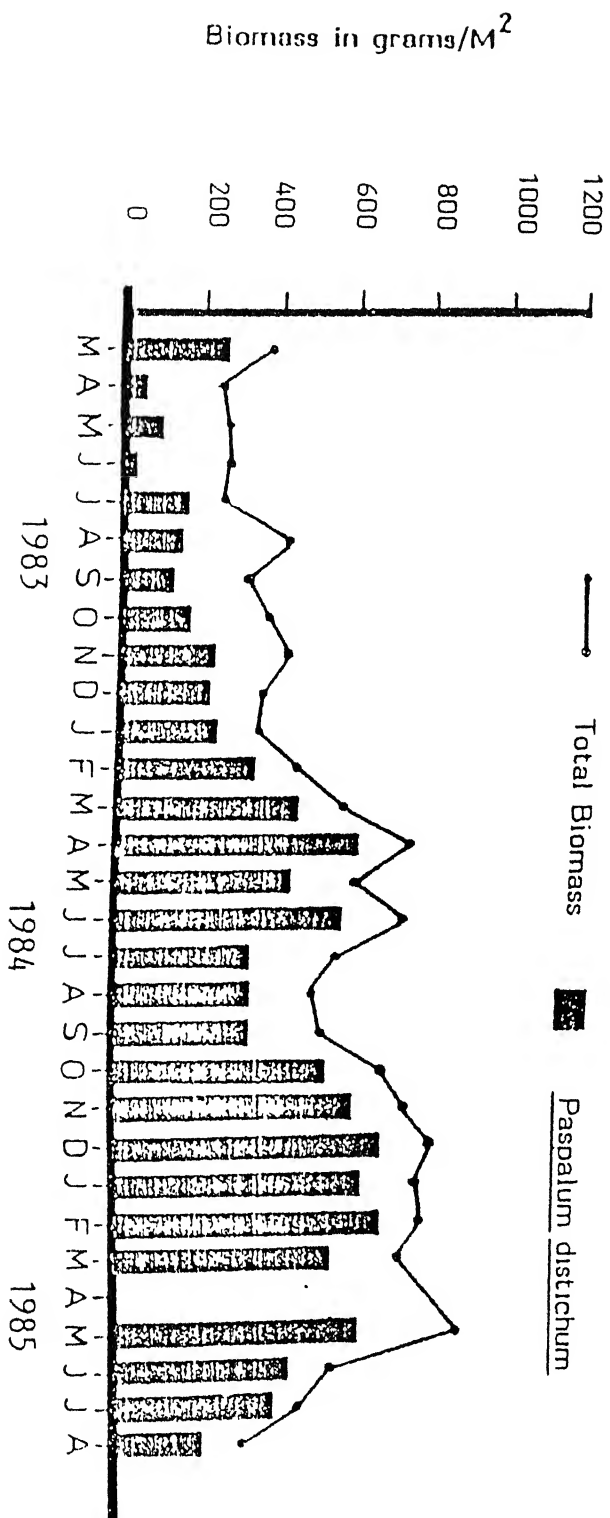
Biomass in grams/M^2

Total Biomass ———

Paspalum distichum ▓

Fig. 9.1 Biomass of aquatic macrophytes during 1983-85 (reprinted from Ali and Vijayan, 1986).

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